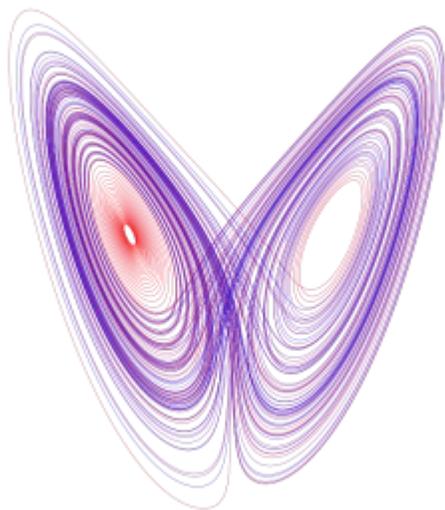


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*Studies in Nonlinear Dynamical
Inference*

**BIFURCATION OF
MACROECONOMETRIC
MODELS AND ROBUSTNESS OF
DYNAMICAL INFERENCE**

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Bifurcation of Macroeconometric Models and Robustness of Dynamical Inferences

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About the Series

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ABSTRACT: In systems theory, it is well known that the parameter spaces of dynamical systems are stratified into bifurcation regions, with each supporting a different dynamical solution regime. Some can be stable, with different characteristics, such as monotonic stability, periodic damped stability, or multiperiodic damped stability, and some can be unstable, with different characteristics, such as periodic, multiperiodic, or chaotic unstable dynamics. But in general the existence of bifurcation boundaries is normal and should be expected from most dynamical systems, whether linear or nonlinear. Bifurcation boundaries in parameter space are not evidence of model defect. While existence of such bifurcation boundaries is well known in economic theory, econometricians using macroeconomic models rarely take bifurcation into consideration, when producing policy simulations from macroeconomics models. Such models are routinely simulated only at the point estimates of the models' parameters.

Barnett and He (1999) explored bifurcation stratification of Bergstrom and Wymer's (1976) continuous time UK macroeconomic model. Bifurcation boundaries intersected the confidence region of the model's parameter estimates. Since then, Barnett and his coauthors have been conducting similar studies of many other newer macroeconomic models spanning all basic categories of those models. So far, they have not found a single case in which the model's parameter space was not subject to bifurcation stratification. In most cases, the confidence region of the parameter estimates were intersected by some of those bifurcation boundaries. The most fundamental implication of this research is that policy simulations with macroeconomic models should be conducted at multiple settings of the parameters within the confidence region. While this result would be as expected by systems theorists, the result contradicts the normal procedure in macroeconomics of conducting policy simulations solely at the point estimates of the parameters.

This survey provides an overview of the classes of macroeconomic models for which these experiments have so far been run and emphasizes the implications for lack of robustness of conventional dynamical inferences from macroeconomic policy simulations. By making this detailed survey of past bifurcation experiments available, we hope to encourage and facilitate further research on this problem with other models and to emphasize the need for simulations at various points within the confidence regions of macroeconomic models, rather than at only point estimates.

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1. Bifurcation of Macroeconomic Models¹

1.1. Introduction

Bifurcation has long been a topic of interest in dynamical macroeconomic systems. Bifurcation analysis is important in understanding dynamic properties of macroeconomic models as well as in selection of stabilization policies. The goal of this survey is to summarize work by William A Barnett and his coauthors on bifurcation analyses in macroeconomic models to facility and motivate work by others on further models. In section 1, we introduce the concept of bifurcation and its role in studies of macroeconomic systems and also discuss several types of bifurcations by providing examples summarized from Barnett and He (2004, 2006b). In sections 2-8, we discuss bifurcation analysis and approaches with models from Barnett's other papers on this subject.

To explain what bifurcation is, Barnett and He (2004,2006b) begin with the general form of many existing macroeconomic models:

$$\mathbf{D}\mathbf{x} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}), \tag{1.1}$$

where \mathbf{D} is the vector-valued differentiation operator, \mathbf{x} is the state vector, $\boldsymbol{\theta}$ is the parameter vector, and \mathbf{f} is the vector of functions governing the dynamics of the system, with each component assumed to be smooth in a local region of interest.

In system (1.1), the focus of interest lies in the settings of the parameter vector, $\boldsymbol{\theta}$. Assume $\boldsymbol{\theta}$ takes values within a theoretically feasible set Θ . The value of $\boldsymbol{\theta}$ can affect the dynamics of the system substantially through a small change, and we say a bifurcation occurs in the system, if such a small change in parameters fundamentally alters the nature of the dynamics of the system. In particular, bifurcation refers to a change in qualitative features instead of quantitative features of the solution dynamics. A change in quantitative features of dynamical solutions may refer to a change in such properties as the period or amplitude of cycles, while a change in qualitative features may refer to such changes as changes from one type of stability or instability to another type of stability or instability.

¹ This section is summarized from Barnett and He (2004,2006b).

A point within the parameter space at which a change in qualitative features of the dynamical solution path occurs defines a point on a bifurcation boundary. At the bifurcation point, the structure of the dynamic system may change fundamentally. Different dynamical solution properties can occur when parameters are close to but on different sides of a bifurcation boundary. A parameter set can be stratified by bifurcation boundaries into several subsets with different types of dynamics within each subset.

There are several types of bifurcation boundaries, such as Hopf, pitchfork, saddle-node, transcritical, and singularity bifurcation. Each type of bifurcation produces a different type of qualitative dynamic change. We illustrate these different types of bifurcation by providing examples in section 1.3. Bifurcation boundaries have been discovered in many macroeconomic systems. For example, Hopf bifurcations have been found in growth models (e.g., Benhabib and Nishimura (1979), Boldrin and Woodford (1990), Dockner and Feichtinger (1991), and Nishimura and Takahashi (1992)) and in overlapping generations models. Pitchfork bifurcations have been found in the tatonnement process (e.g., Bala (1997) and Scarf (1960)). Transcritical bifurcations have been found in Bergstrom and Wymer's (1976) UK model (Barnett and He (1999)) and singularity bifurcation in Leeper and Sims' Euler-equation model (Barnett and He (2008)).

One reason we are concerned about bifurcation phenomena in macroeconomic models is because changes in parameters could affect dynamic behaviors of the models and consequently the outcomes of imposition of policy rules. For example, Bergstrom and Wymer's (1976) UK model operates close to bifurcation boundaries between stable and unstable regions of the parameter space. In this case, if a bifurcation boundary intersects the confidence region of the parameter estimates, different qualitative properties of solution can exist within this confidence region. As a result, robustness of inferences about dynamics can be damaged, especially if inferences about dynamics are based on model simulations with the parameters set only at their point estimates. When confidence regions are stratified by bifurcation boundaries, dynamical inferences need to be based on simulations at points within each of the stratified subsets of the confidence region.

Knowledge of bifurcation boundaries is directly useful in policy selection. If the system is unstable, a successful policy would bifurcate the system from the unstable to stable region.

In that sense, stabilization policy can be viewed as bifurcation selection. As illustrated in section 2, Barnett and He (2002) have shown that successful bifurcation policy selection can be difficult to design.

Barnett's work has found bifurcation phenomena in every macroeconomic model that he and his coauthors have so far explored. Barnett and He (1999,2002) examined the dynamics of Bergstrom-Wymer's continuous-time dynamic macroeconomic model of the UK economy and found both transcritical and Hopf bifurcation boundaries. Barnett and He (2008) estimated and displayed singularity bifurcation boundaries for the Leeper and Sims (1994) Euler equations model. Barnett and Duzhak (2010) found Hopf and period doubling bifurcations in a New Keynesian model. Banerjee, Barnett, Duzhak and Gopalan (2011) examined the possibility of cyclical behavior in the Marshallian Macroeconomic Model. Barnett and Eryilmaz (2013, 2014) investigated bifurcation in open economy models. Barnett and Ghosh (2013a) investigated the existence of bifurcations in endogenous growth models.

This survey is organized in the chronological order of Barnett's work on bifurcation of macroeconomic models, from early models to many of the most recent models.

1.2. Stability

There are two possible approaches to analyze bifurcation phenomena: global and local. Methods in Barnett's current papers have used local analysis, which is analysis of the linearized dynamic system in a neighborhood of the steady state. In his papers, equation (1.1) is linearized in the form

$$\mathbf{D}\mathbf{x} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta}), \quad (1.2)$$

where $\mathbf{A}(\boldsymbol{\theta})$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x}, \boldsymbol{\theta})$, and $\mathbf{F}(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{A}(\boldsymbol{\theta})\mathbf{x} = \mathbf{o}(\mathbf{x}, \boldsymbol{\theta})$ is the vector of higher order term. Define \mathbf{x}^* to be the system's steady state equilibrium, such that $\mathbf{f}(\mathbf{x}^*, \boldsymbol{\theta}) = \mathbf{0}$, and redefine the variables such that the steady state is the point $\mathbf{x}^* = \mathbf{0}$ by replacing \mathbf{x} with $\mathbf{x} - \mathbf{x}^*$.

The local stability of (1.1), for small perturbation away from the equilibrium, can be studied through the eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$, which is a matrix-valued function of the parameters $\boldsymbol{\theta}$. It is important to know at what parameter values, $\boldsymbol{\theta}$, the system (1.1) is unstable. But it is

also important to know the nature of the instability, such as periodic, multiperiodic, or chaotic, and the nature of the stability, such as monotonically convergent, damped single-periodic convergent, or damped multiperiodic convergent. For global analysis, which can be far more complicated than local analysis, higher order terms must be considered, since the perturbations away from the equilibrium can be large. Analysis of $\mathbf{A}(\boldsymbol{\theta})$ alone may not be adequate. More research on global analysis of macroeconomic models is needed.

To analyze the local stability properties of the system, we need to locate the bifurcation boundaries. The boundaries must satisfy

$$\det(\mathbf{A}(\boldsymbol{\theta})) = 0. \quad (1.3)$$

According to Barnett and He (2004), if all eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ have strictly negative real parts, then (1.1) is locally asymptotically stable in the neighborhood of $\mathbf{x} = \mathbf{0}$. If at least one of the eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ has positive real part, then (1.1) is locally asymptotically unstable in the neighborhood of $\mathbf{x} = \mathbf{0}$.

The bifurcation boundaries can be difficult to locate. In Barnett and He (1999, 2002), various methods are applied to locate the bifurcation boundaries characterized by (1.3). Equation (1.3) usually cannot be solved in closed form, when $\boldsymbol{\theta}$ is multi-dimensional. As a result, numerical methods are extensively used for solving (1.3).

Before proceeding to the next section, we introduce the definition of hyperbolic for flows and maps, respectively. According to Hale and Kocak (1991), the following definitions apply.

Definition 1.1. An equilibrium point \mathbf{x}^* of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is said to be *hyperbolic*, if all the eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{x}^*)$ have nonzero real parts.

Definition 1.2. A fixed point \mathbf{x}^* of $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ is said to be *hyperbolic*, if the linear C^1 map $\mathbf{x} \mapsto D\mathbf{f}(\mathbf{x}^*)\mathbf{x}$ is *hyperbolic*; that is, if the Jacobian matrix $D\mathbf{f}(\mathbf{x}^*)$ at \mathbf{x}^* has no eigenvalues with modulus one.

Definition 1.2. refers to discrete-time dynamical systems. Since bifurcations can only occur in a local neighborhood of non-hyperbolic equilibria, we are more interested in the behavior at non-hyperbolic equilibria.

For a discrete-time dynamical system, consider a generic smooth one-parameter family of maps $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \alpha) = \mathbf{f}_{(\alpha)}(\mathbf{x})$, $\mathbf{x} \in R^n, \alpha \in R$. Since local bifurcation happens only at nonhyperbolic fixed points, there are three critical cases to consider:

- (a) The fixed point \mathbf{x}^* has eigenvalue 1.
- (b) The fixed point \mathbf{x}^* has eigenvalue -1.
- (c) The fixed point \mathbf{x}^* has a pair of complex-conjugate eigenvalues $e^{\pm i\theta_0}$ with $0 < \theta_0 < \pi$.

The codimension 1 bifurcation associated with case (a) is called a *fold (saddle node)* bifurcation. The codimension 1 bifurcation associated with case (b) is called a *flip (period doubling)* bifurcation, while the codimension 1 bifurcation associated with case (c) is called a *Neimark-Sacker* bifurcation. *Neimark-Sacker* bifurcation is the equivalent of *Hopf* bifurcation for maps.

In the following section, we are going to introduce three important one-dimensional equilibrium bifurcations described locally by ordinary differential equations. They are transcritical, pitchfork, and saddle-node bifurcations.

1.3. Types of Bifurcations

1.3.1. Transcritical Bifurcations

For a one-dimensional system,

$$Dx = G(x, \theta),$$

the transversality conditions for a transcritical bifurcation at $(x, \theta) = (0, 0)$ are

$$G(0,0) = G_x(0,0) = G_\theta(0,0) = 0, \quad G_{xx}(0,0) \neq 0, \quad \text{and} \quad G_{\theta x}^2 - G_{xx}G_{\theta\theta}(0,0) > 0. \quad (1.4)$$

An example of such a form is

$$Dx = \theta x - x^2. \quad (1.5)$$

The steady state equilibria of the system are at $x^* = 0$ and $x^* = \theta$. It follows that system (1.5) is stable around the equilibrium $x^* = 0$ for $\theta < 0$, and unstable for $\theta > 0$. System (1.5) is stable around the equilibrium $x^* = \theta$ for $\theta > 0$, and unstable for $\theta < 0$. The nature of the dynamics changes as the system bifurcates at the origin. This transcritical bifurcation arises in systems in which there is a simple solution branch, corresponding here to $x^* = 0$.

Transcritical bifurcations have been found in high-dimensional continuous-time macroeconomic systems, but in high dimensional cases, transversality conditions have to be verified on a manifold. Details are provided in Guckenheimer and Holmes (1983).

1.3.2. Pitchfork Bifurcations

For a one-dimensional system,

$$Dx = f(x, \theta).$$

Suppose that there exists an equilibrium x^* and a parameter value θ^* such that (x^*, θ^*) satisfies the following conditions:

$$(a) \quad \frac{\partial f(x, \theta^*)}{\partial x} \Big|_{x=x^*} = 0,$$

$$(b) \quad \frac{\partial^3 f(x, \theta^*)}{\partial x^3} \Big|_{x=x^*} \neq 0,$$

$$(c) \quad \frac{\partial^2 f(x, \theta)}{\partial x \partial \theta} \Big|_{x=x^*, \theta=\theta^*} \neq 0,$$

then (x^*, θ^*) is a pitchfork bifurcation point.

An example of such form is

$$Dx = \theta x - x^3.$$

The steady state equilibria of the system are at $x^* = 0$ and $x^* = \pm\sqrt{\theta}$. It follows that the system is stable when $\theta < 0$ at the equilibrium $x^* = 0$, and unstable at this point when $\theta > 0$. The two other equilibria $x^* = \pm\sqrt{\theta}$ are stable for $\theta > 0$. The equilibrium $x^* = 0$ loses stability, and two new stable equilibria appear. This pitchfork bifurcation, in which a stable

solution branch bifurcates into two new equilibria as θ increases, is called a supercritical bifurcation.

Bala (1997) shows how pitchfork bifurcation can occur in the tatonnement process.

1.3.3. Saddle-Node Bifurcations

For a one-dimensional system,

$$Dx = f(x, \theta).$$

A saddle-node point (x^*, θ^*) satisfies the equilibrium condition $f(x^*, \theta^*) = 0$ and the Jacobian condition $\frac{\partial f(x, \theta)}{\partial x} \Big|_{x=x^*} = 0$, as well as the transversality conditions, as follows:

$$(a) \quad \frac{\partial f(x, \theta)}{\partial \theta} \Big|_{x=x^*, \theta=\theta^*} \neq 0,$$

$$(b) \quad \frac{\partial^2 f(x, \theta)}{\partial x^2} \Big|_{x=x^*, \theta=\theta^*} \neq 0.$$

Sotomayor (1973) shows that transversality conditions for high-dimensional systems can also be formulated.

A simple system with a saddle-node bifurcation is

$$Dx = \theta - x^2.$$

The equilibria are at $x^* = \pm\sqrt{\theta}$, which requires θ to be nonnegative. Therefore, there exist no equilibria for $\theta < 0$, and there exist two equilibria at $x^* = \pm\sqrt{\theta}$, when $\theta > 0$. It follows that when $\theta > 0$, the system is stable at $x^* = \sqrt{\theta}$ and unstable at $x^* = -\sqrt{\theta}$. In this example, bifurcation occurs at the origin as θ increases through zero, which is called the (supercritical) saddle node.

1.3.4. Hopf Bifurcations

Hopf bifurcation is the most studied type of bifurcation in economics. For continuous time systems, Hopf bifurcation occurs at the equilibrium points at which the system has a

Jacobian matrix with a pair of purely imaginary eigenvalues and no other eigenvalues which have zero real parts. For discrete time system, the following theorem applies in the special case of $n=2$. The Hopf Bifurcation Theorem in Gandolfo (2010, ch. 24, p.497) is widely applied to find the existence of Hopf bifurcation.

Theorem 1.1. (*Existence of Hopf Bifurcation in 2 dimensions*) Consider the two-dimensional non-linear difference system with one parameter

$$\mathbf{y}_{t+1} = \boldsymbol{\varphi}(\mathbf{y}_t, \alpha),$$

and suppose that for each α in the relevant interval there exists a smooth family of equilibrium points, $\mathbf{y}_e = \mathbf{y}_e(\alpha)$, at which the eigenvalues are complex conjugates, $\lambda_{1,2} = \theta(\alpha) + i\omega(\alpha)$. If there is a critical value α_0 of the parameter such that

- a. the eigenvalues' modulus becomes unity at α_0 , but the eigenvalues are not roots of unity (from the first up to the fourth), namely

$$|\lambda_{1,2}(\alpha_0)| = \sqrt{\theta^2 + \omega^2} = 1, \quad \lambda_{1,2}^j(\alpha_0) \neq 1 \text{ for } j = 1, 2, 3, 4,$$

and

- b. $\left. \frac{d|\lambda_{1,2}(\alpha)|}{d\alpha} \right|_{\alpha=\alpha_0} \neq 0,$

then there is an invariant closed curve bifurcating from α_0 .

This theorem only applies with a 2x2 Jacobian. The earliest theoretical works on Hopf bifurcation include Poincaré (1892) and Andronov (1929), both of which were concerned with two-dimensional vector fields. A general theorem on the existence of Hopf bifurcation, which is valid in n dimensions, was proved by Hopf (1942).

A simple example in the two-dimensional system is

$$Dx = -y + x(\theta - (x^2 + y^2)),$$

$$Dy = x + y(\theta - (x^2 + y^2)).$$

One equilibrium is $x^* = y^* = 0$ with stability occurring for $\theta < 0$ and the instability occurring for $\theta > 0$. That equilibrium has a pair of conjugate eigenvalues $\theta + i$ and $\theta - i$. The eigenvalues become purely imaginary, when $\theta = 0$.

Barnett and He (2004) show the following method to find Hopf bifurcation. They let $p(s) = \det(s\mathbf{I} - \mathbf{A})$ be the characteristic polynomial of \mathbf{A} and write it as

$$p(s) = c_0 + c_1s + c_2s^2 + c_3s^3 + \dots + c_{n-1}s^{n-1} + s^n.$$

They construct the following $(n - 1)$ by $(n - 1)$ matrix

$$\mathbf{S} = \begin{bmatrix} c_0 & c_2 & \dots & c_{n-2} & 1 & 0 & 0 & \dots & 0 \\ 0 & c_0 & c_2 & \dots & c_{n-2} & 1 & 0 & \dots & 0 \\ & & & & \dots & & & & \\ 0 & 0 & \dots & 0 & c_0 & c_2 & c_4 & \dots & 1 \\ c_1 & c_3 & \dots & c_{n-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & c_1 & c_3 & \dots & c_{n-1} & 0 & 0 & \dots & 0 \\ & & & & \dots & & & & \\ & & & & \dots & & & & \\ 0 & 0 & \dots & 0 & c_1 & c_3 & \dots & \dots & c_{n-1} \end{bmatrix}.$$

Let \mathbf{S}_0 be obtained by deleting rows 1 and $\frac{n}{2}$ and columns 1 and 2, and let \mathbf{S}_1 be obtained by deleting rows 1 and $\frac{n}{2}$ and columns 1 and 3. The matrix $\mathbf{A}(\theta)$ has one pair of purely imaginary eigenvalues (Guckenheimer, Myers, and Sturmfels (1997)), if

$$\det(\mathbf{S}) = 0, \quad \det(\mathbf{S}_0) \det(\mathbf{S}_1) > 0. \quad (1.6)$$

If $\det(\mathbf{S}) = 0$ and $\det(\mathbf{S}_0) \det(\mathbf{S}_1) = 0$, then $\mathbf{A}(\theta)$ may have more than one pair of purely imaginary eigenvalues. The following condition can be used to find candidates for bifurcation boundaries:

$$\det(\mathbf{S}) = 0, \quad \det(\mathbf{S}_0) \det(\mathbf{S}_1) \geq 0. \quad (1.7)$$

Since solving (1.7) analytically is difficult, Barnett and He (1999) apply the following numerical procedure to find bifurcation boundaries. Without loss of generality, they initially consider only two parameters θ_1 and θ_2 .

Procedure (P1)

- (1) For any fixed θ_1 , treat θ_2 as a function of θ_1 , and find the value of θ_2 satisfying the condition $h(\theta_2) = \det(\mathbf{A}(\theta)) = 0$. First find the number of zeros of $h(\theta_2)$. Starting with approximations of zeros, use the following gradient algorithm to find all zeros of $h(\theta_2)$:

$$\theta_2(n+1) = \theta_2(n) - a_n h(\theta_2) \Big|_{\theta_2=\theta_2(n)} \quad (1.8)$$

where $\{a_n, n = 0, 1, 2, \dots\}$ is a sequence of positive step sizes.

- (2) Repeat the same procedure to find all θ_2 satisfying (1.7).
- (3) Plot all the pairs (θ_1, θ_2) .
- (4) Check all parts of the plot to find the segments representing the bifurcation boundaries. Then parts of the curve found in step (1) are boundaries of saddle-node bifurcations. Parts of the curve found in step (2) are boundaries of Hopf bifurcations, if the required transversality conditions are satisfied.

Pioneers in studies of Hopf bifurcations in economics include Torre (1977) and Benhabib and Nishimura (1979). Torre found the appearance of a limit cycle associated with a Hopf bifurcation boundary in Keynesian systems. Benhabib and Nishimura showed that a closed invariant curve might emerge as the result of optimization in a multi-sector neoclassical optimal growth model. These studies illustrate the existence of a Hopf bifurcation boundary in an economic model results in a solution following closed curves around the stationary state. The solution paths may be stable or unstable, depending upon the side of the bifurcation boundary on which the parameter values lie. More recent studies finding Hopf bifurcation in econometric models include Barnett and He (1999, 2002, 2008), who found bifurcation boundaries of the Bergstrom-Wymer continuous-time UK model and the Leeper and Sims Euler-equations model.

1.3.5. Singularity-Induced Bifurcations

This section is devoted to a dramatic kind of bifurcation found by Barnett and He (2008) in the Leeper and Sims (1977) model—singularity-induced bifurcation.

Some macroeconomic models, such as the dynamic Leontief model (Luenberger and Arbel (1977)) and the Leeper and Sims (1994) model, have the form

$$\mathbf{B}\mathbf{x}(t + 1) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t). \quad (1.9)$$

Here $\mathbf{x}(t)$ is the state vector, $\mathbf{f}(t)$ is the vector of driving variables, t is time, and \mathbf{B} and \mathbf{A} are constant matrices of appropriate dimensions. If $\mathbf{f}(t) = \mathbf{0}$, the system (1.9) is in the class of autonomous systems. Barnett and He (2006b) illustrate only the autonomous cases of (1.9).

If \mathbf{B} is invertible, then we can invert \mathbf{B} to acquire

$$\mathbf{x}(t + 1) = \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t),$$

so that

$$\begin{aligned} \mathbf{x}(t + 1) - \mathbf{x}(t) &= \mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t) - \mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t) \\ &= (\mathbf{B}^{-1}\mathbf{A} - \mathbf{I})\mathbf{x}(t) + \mathbf{B}^{-1}\mathbf{f}(t), \end{aligned}$$

which is in the form of (1.1).

The case in which the matrix \mathbf{B} is singular is of particular interest. Barnett and He (2006b) rewrite (1.9) by generalizing the model to permit nonlinearity as follows:

$$\mathbf{B}(\mathbf{x}(t), \boldsymbol{\theta})\mathbf{D}\mathbf{x} = \mathbf{F}(\mathbf{x}(t), \mathbf{f}(t), \boldsymbol{\theta}). \quad (1.10)$$

Here $\mathbf{f}(t)$ is the vector of driving variables, and t is time. Barnett and He (2006b) consider the autonomous cases in which $\mathbf{f}(t) = \mathbf{0}$.

Singularity-induced bifurcation occurs, when the rank of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ changes, as from an invertible matrix to a singular one. Therefore, the matrix must depend on $\boldsymbol{\theta}$ for such changes to occur. If the rank of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ does not change according to the change of $\boldsymbol{\theta}$, then singularity of $\mathbf{B}(\mathbf{x}, \boldsymbol{\theta})$ is not sufficient for (1.10) to be able to produce singularity bifurcation.

Barnett and He (2006b) consider the two-dimensional state-space case and perform an appropriate coordinate transformation allowing (1.10) to become the following equivalent form:

$$\mathbf{B}_1(x_1, x_2, \boldsymbol{\theta}) D\mathbf{x}_1 = \mathbf{F}_1(x_1, x_2, \boldsymbol{\theta}),$$

$$0 = \mathbf{F}_2(x_1, x_2, \boldsymbol{\theta}).$$

They provide four examples to demonstrate the complexity of bifurcation behaviors that can be produced from system (1.10). The first two examples do not produce singularity bifurcations, since \mathbf{B} does not depend on $\boldsymbol{\theta}$. In the second two examples, Barnett and He (2008) find singularity bifurcation, since \mathbf{B} does depend on $\boldsymbol{\theta}$.

Example 1. Consider the following system modified from system (1.5), which has been shown to produce transcritical bifurcation:

$$Dx = \theta x - x^2, \tag{1.11}$$

$$0 = x - y^2. \tag{1.12}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is singular but does not depend upon the value of θ .

The equilibria are $(x^*, y^*) = (0,0)$ and $(\theta, \pm\sqrt{\theta})$. Near the equilibrium $(x^*, y^*) = (0,0)$, the system ((1.11),(1.12)) is stable for $\theta < 0$ and unstable for $\theta > 0$. The equilibria $(x^*, y^*) = (\theta, \pm\sqrt{\theta})$ are undefined, when $\theta < 0$, and stable when $\theta > 0$. The bifurcation point is $(x, y, \theta) = (0,0,0)$. Notice before and after bifurcation, the number of differential equations and the number of algebraic equations remain unchanged. This implies that the bifurcation point does not produce singularity bifurcation, since \mathbf{B} does not depend upon θ .

Example 2. Consider the following system modified from system (1.7), which can produce saddle-node bifurcation:

$$Dx = \theta - x^2, \tag{1.13}$$

$$0 = x - y^2. \tag{1.14}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is singular but does not depend upon the value of θ .

The equilibria are at $(x^*, y^*) = (\sqrt{\theta}, \pm^4\sqrt{\theta})$, defined only for $\theta \geq 0$. The system ((1.13),(1.14)) is stable around both of the equilibria $(x^*, y^*) = (\sqrt{\theta}, \pm^4\sqrt{\theta})$ and $(x^*, y^*) = (\sqrt{\theta}, \pm^4\sqrt{\theta})$. The bifurcation point is $(x^*, y^*, \theta) = (0,0,0)$. The three-dimensional bifurcation diagram in Barnett and He (2006b) shows that there is no discontinuity or change in dimension at the origin at the origin. The bifurcation point does not produce singularity bifurcation, since the dimension of the state space dynamics remains unchanged on either side of the origin.

Example 3. Consider the following system:

$$Dx = ax - x^2, \text{ with } a > 0, \tag{1.15}$$

$$\theta Dy = x - y^2. \tag{1.16}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix},$$

which does depend upon the parameter θ .

When $\theta = 0$, the system has one differential equation (1.15) and one algebraic equation (1.16). If $\theta \neq 0$, the system has two differential equations (1.15) and (1.16) with no algebraic equations for nonzero θ .

The equilibria are $(x^*, y^*) = (0,0)$ and $(a, \pm\sqrt{a})$. For any value of θ , the system ((1.15),(1.16)) is unstable around the equilibrium at $(x^*, y^*) = (0,0)$. The equilibrium $(x^*, y^*) = (a, \sqrt{a})$ is unstable for $\theta < 0$ and stable for $\theta > 0$. The equilibrium $(x^*, y^*) = (a, -\sqrt{a})$ is unstable for $\theta > 0$ and stable for $\theta < 0$.

Without loss of generality, Barnett and He (2006b) normalize a to be 1. When $\theta = 0$, the system's behavior degenerates into movement along the one-dimensional curve $x - y^2 =$

0. When $\theta \neq 0$, the dynamics of the system move throughout the two-dimensional state space. The singularity bifurcation caused by the transition from nonzero θ to zero results in the drop in the dimension.

Barnett and He (2006b) observe that even if singularity bifurcation does not cause a change of the system between stability and instability, dynamical properties produced by singularity bifurcation can change. For example, if θ changes from positive to zero, when (x, y) is at the equilibrium $(1,1)$, the system will remain stable; if θ changes from positive to zero, when (x, y) is at the equilibrium $(0,0)$, the system will remain unstable; if θ changes from positive to zero, when (x, y) is at the equilibrium $(1,-1)$, the system will change from unstable to stable. But in all of these cases, the nature of the disequilibrium dynamics changes dramatically, even if there is no transition between stability and instability.

Example 4. Consider the following system:

$$Dx = ax - x^2, \text{ with } a > 0, \tag{1.17}$$

$$\theta Dy = x - y. \tag{1.18}$$

Comparing with the general form of (1.10), observe that

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix}.$$

The equilibria are $(x^*, y^*) = (0,0)$ and (a, a) . The system is unstable around the equilibrium $(x^*, y^*) = (0,0)$ for any value of θ . The equilibrium $(x^*, y^*) = (a, a)$ is unstable for $\theta < 0$ and stable for $\theta \geq 0$. When $\theta < 0$, the system is unstable everywhere. When $\theta = 0$, equation (1.18) becomes the algebraic constraint $y = x$, which is a one-dimensional ray through the origin. However, when $\theta \neq 0$, the system moves into the two-dimensional space. Even though the dimension can drop from singular bifurcation, there could be no change between stability and instability. For example, $(0,0)$ remains unstable and $(1, 1)$ remains stable, when $\theta \neq 0$ and $\theta = 0$.

Barnett and He (2006b) also observe that the nature of the dynamics with θ small and positive is very different from the dynamics with θ small and negative. In particular, the

equilibrium at $(x^*, y^*) = (1, 1)$ is stable in the former case and unstable in the latter case. Hence there is little robustness of dynamical inference to small changes of θ close to the bifurcation boundary. Barnett and Binner (2004, part 4) further investigate the subject of robustness of inferences in dynamic models.

Example 5. Consider the following system:

$$\begin{aligned} Dx_1 &= x_3, \\ Dx_2 &= -x_2, \\ 0 &= x_1 + x_2 + \theta x_3, \end{aligned} \tag{1.19}$$

with singular matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.20}$$

where $\mathbf{D}\mathbf{x} = (Dx_1, Dx_2, Dx_3)'$.

The only equilibrium is at $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (0, 0, 0)$. For any $\theta \neq 0$, Barnett and He (2006b) solve the last equation for x_3 and substitute into the first equation to derive the following two equation system:

$$Dx_1 = -\frac{x_1 + x_2}{\theta}, \tag{1.21}$$

$$Dx_2 = -x_2.$$

In this case, the matrix \mathbf{B} becomes the identity matrix.

This two-dimensional system is stable at $\mathbf{x}^* = (x_1^*, x_2^*) = (0, 0)$ for $\theta > 0$ and unstable for $\theta < 0$. However, setting $\theta = 0$, Barnett and He (2006b) find that system (1.19) becomes

$$x_1 = -x_2,$$

$$Dx_2 = -x_2,$$

$$x_3 = x_2, \tag{1.22}$$

for all $t > 0$. This system has the following singular matrix :

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.23}$$

The dimension of system (1.22) is very different from that of (1.21). In system (1.22), there are two algebraic constraints and one differential equation, while system (1.21) has two differential equations and no algebraic constraints. Clearly the matrix \mathbf{B} is different in the two cases with different ranks. This example shows that singular bifurcation can result from the dependence of \mathbf{B} upon the parameters, even if there does not exist a direct closed-form algebraic representation of the dependence.

Barnett and He (2008) find singularity bifurcation in their research on the Leeper and Sims' Euler-equations macroeconomic model, as surveyed in section 3. Singularity bifurcations could similarly damage robustness of dynamic inferences with other modern Euler-equations macroeconomic models. Examples above show that implicit function systems (1.9) and (1.10) could produce singular bifurcation, while closed form differential equations systems are less likely to produce singularity bifurcation. Since Euler equation systems are in implicit function form and rarely can be solved for closed form representations, Barnett and He (2006b) conclude that singularity bifurcation should be a serious concern with modern Euler equations models.

2. Bergstrom—Wymer Continuous Time UK Model²

2.1. Introduction

Among the models that have direct relevance to this research are the high dimensional continuous time macroeconomic models in Bergstrom, Nowman and Wymer (1992), Bergstrom, Norman, and Wandasiewicz (1994), Bergstrom and Wymer (1976), Grandmont (1998), Leeper and Sims (1994), Powell and Murphy (1997), and Kim (2000). Surveys of

² This section is based on Barnett and He (1999,2001b,2002).

macroeconomic models are available in Bergstrom (1996) and in several textbooks such as Gandolfo (1996) and Medio (1992). The general theory of economic dynamics is provided, for example, in Boldrin and Woodford (1990) and Gandolfo (1992). Various bifurcation phenomena are reported in Bala (1997), Benhabib (1979), Medio (1992), Gandolfo (1992), and Nishimura and Takahashi (1992). Focused studies of stability are conducted in Grandmont (1998), Scarf (1960), and Nieuwenhuis and Schoonbeek (1997). Barnett and Chen (1988) discovered chaotic behaviors in economics. Bergstrom, Nowman, and Wandasiewicz (1994) investigate stabilization of macroeconomic models using policy control. Wymer (1997) describes several mathematical frameworks for the study of structural properties of macroeconomic models.

In section 2, we discuss several papers by Barnett and He on bifurcation analysis using Bergstrom, Nowman, and Wymer's continuous-time dynamic macroeconomic model of the UK economy. Barnett and He chose this policy-relevant model as their first to try, because the model is particularly well suited to these experiments. The model contains adjustment speeds producing Keynesian rigidities and hence possible Pareto improving policy intervention. In addition, as a system of second order differential equations, the model can produce interesting dynamics and possesses enough equations and parameters to be fitted plausibly to the UK data.

Barnett and He (1999) discovered that both saddle-node bifurcations and Hopf bifurcations coexist within the model's region of plausible parameter setting. Bifurcation boundaries are located and drawn. The model's Hopf bifurcation helps to provide explanations for some cyclical phenomena in the UK macroeconomy. The Barnett and He paper designed a numerical algorithm for locating the model's bifurcation boundaries. That algorithm was provide above in section 1.3.4.

Barnett and He (1999) observed that stability of the model had not previously been tested. They found that the point estimates of the model's parameters are outside the stable subset of the parameter space, but close enough to the bifurcation boundary so that the hypothesis of stability cannot be rejected. Confidence regions around the parameter estimates are intersected by the boundary separating stability from instability, with the point estimates being on the unstable side.

Barnett and He (2002) explored the problem of selection of a “stabilization policy.” The purpose of the policy was to bifurcate the system from an unstable to a stable operating regime by moving the parameters’ point estimates into the stable region. The relevant parameter space is the augmented parameter space, including both the private sector’s parameters and the parameters of the policy rule. Barnett and He found that policies producing successful bifurcation to stability are difficult to determine, and the policies recommended by the originators of the model, based on reasonable economic intuition and full knowledge of their own model, tend to be counterproductive, since such policies contract the size of the stable subset of the parameter space and move that set farther away from the private sector’s parameter estimates. These results point towards the difficulty of designing successful countercyclical stabilization policy in the real world, where the structure of the economy is not accurately known. Barnett and He (1999) also proposed a new formula for determining the bifurcation boundaries for transcritical bifurcations.

2.2. The Model³

The Bergstrom, Nowman, and Wymer (1992) model is described by the following 14 second-order differential equations.

$$D^2 \log C = \gamma_1(\lambda_1 + \lambda_2 - D \log C) + \gamma_2 \log \left[\frac{\beta_1 e^{-\{\beta_2(r-D \log p) + \beta_3 D \log p\}} (Q+P)}{T_1 C} \right] \quad (2.1)$$

$$D^2 \log L = \gamma_3(\lambda_2 - D \log L) + \gamma_4 \log \left[\frac{\beta_4 e^{-\lambda_4 t} \{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{L} \right] \quad (2.2)$$

$$D^2 \log K = \gamma_3(\lambda_1 + \lambda_2 - D \log K) + \gamma_6 \log \left[\frac{\beta_5 \left(\frac{Q}{K}\right)^{1+\beta_6}}{r - \beta_7 D \log p + \beta_8} \right] \quad (2.3)$$

³ The model description is modified from Barnett and He (1999).

$$D^2 \log Q = \gamma_7(\lambda_1 + \lambda_2 - D \log Q) + \gamma_8 \log \left[\frac{\{1 - \beta_9 (\frac{qP}{P_i})^{\beta_{10}}\} (C + G_c + DK + E_n + E_o)}{Q} \right] \quad (2.4)$$

$$D^2 \log p = \gamma_9 \left(D \log \left(\frac{w}{p} \right) - \lambda_1 \right) + \gamma_{10} \log \left[\frac{\beta_{11} \beta_4 T_2 w e^{-\lambda_1 t} \{1 - \beta_5 (\frac{Q}{K})^{\beta_6}\}^{\frac{1+\beta_6}{\beta_6}}}{p} \right] \quad (2.5)$$

$$D^2 \log w = \gamma_{11} \left(\lambda_1 - D \log \left(\frac{w}{p} \right) \right) + \gamma_{12} D \log \left(\frac{P_i}{qP} \right) + \gamma_{13} \log \left[\frac{\beta_4 e^{-\lambda_1 t} \{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{\frac{1}{\beta_6}}}{\beta_{12} e^{\lambda_2 t}} \right] \quad (2.6)$$

$$D^2 r = -\gamma_{14} D r + \gamma_{15} \left[\beta_{13} + r_f - \beta_{14} D \log q + \beta_{15} \frac{p(Q+P)}{M} - r \right] \quad (2.7)$$

$$D^2 \log I = \gamma_{16} \left(\lambda_1 + \lambda_2 - D \log \left(\frac{pI}{qP} \right) \right) + \gamma_{17} \log \left[\frac{\beta_9 (\frac{qP}{P_i})^{\beta_{10}} (C + G_c + DK + E_n + E_o)}{(\frac{P_i}{qP}) I} \right] \quad (2.8)$$

$$D^2 \log E_n = \gamma_{18} (\lambda_1 + \lambda_2 - D \log E_n) + \gamma_{19} \log \left[\frac{\beta_{16} Y_f^{\beta_{17}} (p_f / qP)^{\beta_{18}}}{E_n} \right] \quad (2.9)$$

$$D^2 F = -\gamma_{20} D F + \gamma_{21} [\beta_{19} (Q + P) - F] \quad (2.10)$$

$$D^2 P = -\gamma_{22} D P + \gamma_{23} \{[\beta_{20} + \beta_{21} (r_f - D \log p_f)] K_a - P\} \quad (2.11)$$

$$D^2 K_a = -\gamma_{24} D K_a + \gamma_{25} \{[\beta_{22} + \beta_{23} (r_f - r) - \beta_{24} D \log q - \beta_{25} d_x] (Q + P) - K_a\} \quad (2.12)$$

$$D^2 \log M = \gamma_{26}(\lambda_3 - D \log M) + \gamma_{27} \log \left[\frac{\beta_{26} e^{\lambda_3 t}}{M} \right] \\ + \gamma_{28} D \log \left[\frac{E_n + E_o + P - F}{(p_i / qp)I} \right] + \gamma_{29} \log \left[\frac{E_n + E_o + P - F - DK_a}{(p_i / qp)I} \right] \quad (2.13)$$

$$D^2 \log q = \gamma_{30} D \log(p_f / qp) + \gamma_{31} \log \left[\frac{\beta_{27} p_f}{qp} \right] + \gamma_{32} D \log \left[\frac{E_n + E_o + P - F}{(p_i / qp)I} \right] \\ + \gamma_{33} \log \left[\frac{E_n + E_o + P - F - DK_a}{(p_i / qp)I} \right] \quad (2.14)$$

where t is time, D is the derivative operator, $Dx = dx/dt$, $D^2x = d^2x/dt^2$, and $C, E_n, F, I, K, K_a, L, M, P, Q, q, r, w$ are endogenous variables whose definitions are listed below:

C = real private consumption,

E_n = real non-oil exports,

F = real current transfers abroad,

I = volume of imports,

K = amount of fixed capital,

K_a = cumulative net real investment abroad (excluding changes in official reserve),

L = employment,

M = money supply,

P = real profits, interest and dividends from abroad,

p = price level,

Q = real net output,

q = exchange rate (price of sterling in foreign currency),

r = interest rate,

w = wage rate.

The variables $d_x, E_o, G_c, p_f, p_i, r_f, T_1, T_2, Y_f$ are exogenous variables with the following definitions:

d_x = dummy variables for exchange controls ($d_x = 1$ for 1974-79, $d_x = 0$ for 1980 onwards),

E_o = real oil exports,

G_c = real government consumption,

p_f = price level in leading foreign industrial countries,

p_i = price of imports (in foreign currency),

r_f = foreign interest rate,

T_1 = total taxation policy variable, so $(Q+P)/T_1$ is real private disposable income

T_2 = indirect taxation policy variable so Q/T_2 is real output at factor cost

Y_f = real income of leading foreign industrial countries.

According to Barnett and He (1999), the structural parameters $\beta_i, i = 1, 2, \dots, 27, \gamma_j, j = 1, 2, \dots, 33,$ and $\lambda_k, k = 1, 2, 3,$ can be estimated from historical data. A set of their estimates using quarterly data from 1974 to 1984 are given in Table 2 of Bergstrom, Nowman, and Wymer (1992) and the interpretations of those 14 equations are also available in Bergstrom, Nowman and Wymer (1992).

The exogenous variables satisfy the following conditions in equilibrium:

$$d_x = 0,$$

$$E_o = 0,$$

$$G_c = g^*(Q + P),$$

$$p_f = p_f^* e^{\lambda_4 t},$$

$$p_i = p_i^* e^{\lambda_4 t},$$

$$r_f = r_f^*,$$

$$T_1 = T_1^*,$$

$$T_2 = T_2^*,$$

$$Y_f = Y_f^* e^{\left(\frac{\lambda_1 + \lambda_2}{\beta_{17}}\right)t},$$

where g^* , p_f^* , p_i^* , r_f^* , T_1^* , T_2^* , Y_f^* , and λ_4 are constants. It has been proven that $C(t), \dots, q(t)$ in (2.1)-(2.14) change at constant rates in equilibrium. To study the dynamics of the system around the equilibrium, Barnett and He (2002) make a transformation by defining a set of new variables $y_1(t), y_2(t), \dots, y_{14}(t)$ as follows:

$$y_1(t) = \log\{C(t)/C^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_2(t) = \log\{L(t)/L^* e^{\lambda_2 t}\},$$

$$y_3(t) = \log\{K(t)/K^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_4(t) = \log\{Q(t)/Q^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_5(t) = \log\{p(t)/p^* e^{(\lambda_3 - \lambda_1 - \lambda_2)t}\},$$

$$y_6(t) = \log\{w(t)/w^* e^{(\lambda_3 - \lambda_2)t}\},$$

$$y_7(t) = r(t) - r^*,$$

$$y_8(t) = \log\{I(t)/I^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_9(t) = \log\{E_n(t)/E_n^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_{10}(t) = \log\{F(t)/F^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_{11}(t) = \log\{P(t)/P^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_{12}(t) = \log\{K_a(t)/K_a^* e^{(\lambda_1 + \lambda_2)t}\},$$

$$y_{13}(t) = \log\{M(t)/M^* e^{\lambda_3 t}\},$$

$$y_{14}(t) = \log\{q(t)/q^* e^{(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)t}\},$$

where $C^*, L^*, K^*, Q^*, p^*, w^*, r^*, I^*, E_n^*, F^*, P^*, K_a^*, M^*, q^*$ are functions of the vector (β, γ, λ) of 63 parameters in equations (2.1)-(2.14) and the additional parameters $g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, \lambda_4$.

The following is a set of differential equations derived from (2.1)-(2.14):

$$D^2 y_1 = -\gamma_1 D y_1 + \gamma_2 \{\log(Q^* e^{y_4} + P^* e^{y_{11}}) - \log(Q^* + P^*) - \beta_2 y_7 + (\beta_2 - \beta_3) D y_5 - y_1\} \quad (2.15)$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left\{ \frac{1}{\beta_6} \log \left[\frac{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}}{(Q^*)^{-\beta_6} e^{-\beta_6 y_4} - \beta_5 (K^*)^{-\beta_6} e^{-\beta_6 y_3}} \right] - y_2 \right\} \quad (2.16)$$

$$D^2 y_3 = -\gamma_5 D y_3 + \gamma_6 \{(1 + \beta_6)(y_4 - y_3) + \log[r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8] - \log[y_7 + r^* - \beta_7(D y_5 + \lambda_3 - \lambda_1 - \lambda_2) + \beta_8]\} \quad (2.17)$$

$$D^2 y_4 = -\gamma_7 D y_4 + \gamma_8 \left\{ \log \left[\frac{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}} e^{\beta_{10}(y_5 + y_{14})}}{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}}} \right] + \log(C^* e^{y_1} + g^* (Q^* e^{y_4} + P^* e^{y_{11}}) + K^* e^{y_3} (D y_3 + \lambda_1 + \lambda_2) + E_n^* e^{y_9}) - \log(C^* + g^* (Q^* + P^*) + K^* (\lambda_1 + \lambda_2) + E_n^*) - y_4 \right\} \quad (2.18)$$

$$D^2 y_5 = \gamma_9 (D y_6 - D y_5) + \gamma_{10} \left\{ y_6 - y_5 - \frac{1 + \beta_6}{\beta_6} \log \left[1 - \beta_5 \left(\frac{Q^*}{K^*} \right)^{\beta_6} e^{\beta_6 (y_4 - y_3)} \right] + \frac{1 + \beta_6}{\beta_6} \log \left[1 - \beta_5 \left(\frac{Q^*}{K^*} \right)^{\beta_6} \right] \right\} \quad (2.19)$$

$$D^2 y_6 = \gamma_{11}(Dy_5 - Dy_6) - \gamma_{12}(Dy_5 + Dy_{14}) + \gamma_{13} \left\{ \frac{1}{\beta_6} \log \left[(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6} \right] \right. \\ \left. - \frac{1}{\beta_6} \log \left[(Q^*)^{-\beta_6} e^{-\beta_6 y_4} - \beta_5 (K^*)^{-\beta_6} e^{-\beta_6 y_3} \right] \right\} \quad (2.20)$$

$$D^2 y_7 = -\gamma_{14} Dy_7 + \gamma_{15} \left[\beta_{15} \frac{p^* e^{y_5} (Q^* e^{y_4} + P^* e^{y_{11}})}{M^* e^{y_{13}}} - \beta_{15} \frac{p^* (Q^* + P^*)}{M^*} - \beta_{14} Dy_{14} - y_7 \right] \quad (2.21)$$

$$D^2 y_8 = \gamma_{16}(Dy_5 + Dy_{14} - Dy_8) + \gamma_{17} \{ (1 + \beta_{10})(y_5 + y_{14}) - y_8 \\ + \log[C^* e^{y_1} + g^*(Q^* e^{y_4} + P^* e^{y_{11}}) + K^* e^{y_3} (Dy_3 + \lambda_1 + \lambda_2) + E_n^* e^{y_9}] \\ - \log[C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*] \} \quad (2.22)$$

$$D^2 y_9 = -\gamma_{18} Dy_9 - \gamma_{19} \{ \beta_{18}(y_5 + y_{14}) + y_9 \} \quad (2.23)$$

$$D^2 y_{10} = -\{ \gamma_{20} + 2(\lambda_1 + \lambda_2) \} Dy_{10} - (Dy_{10})^2 + \gamma_{21} \beta_{19} \left\{ \frac{Q^* e^{y_4} + P^* e^{y_{11}}}{F^* e^{y_{10}}} - \frac{Q^* + P^*}{F^*} \right\} \quad (2.24)$$

$$D^2 y_{11} = -\{ \gamma_{22} + 2(\lambda_1 + \lambda_2) \} Dy_{11} - (Dy_{11})^2 + \gamma_{23} \{ \beta_{20} + \beta_{21}(r_f^* - \lambda_4) \} \left[\frac{K_a^* e^{y_{12}}}{P^* e^{y_{11}}} - \frac{K_a^*}{P^*} \right] \quad (2.25)$$

$$D^2 y_{12} = -\{ \gamma_{24} + 2(\lambda_1 + \lambda_2) \} Dy_{12} - (Dy_{12})^2 + \gamma_{25} \{ [\beta_{22} + \beta_{23}(r_f^* - r^* - y_7) \\ - \beta_{24}(Dy_{14} + \lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)] \frac{Q^* e^{y_4} + P^* e^{y_{11}}}{K_a^* e^{y_{12}}} - [\beta_{22} + \beta_{23}(r_f^* - r^*) \\ - \beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)] \frac{Q^* + P^*}{K_a^*} \} \quad (2.26)$$

$$D^2 y_{13} = -\gamma_{26} Dy_{13} - \gamma_{27} y_{13} + \gamma_{28} \left\{ \frac{E_n^* e^{y_9} Dy_9 + P^* e^{y_{11}} Dy_{11} - F^* e^{y_{10}} Dy_{10}}{E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}} \right. \\ \left. + Dy_5 + Dy_{14} - Dy_8 \right\} + \gamma_{29} \{ \log[E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}] \\ - K_a^* e^{y_{12}} (Dy_{12} + \lambda_1 + \lambda_2) \} - \log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)] \\ + y_5 + y_{14} - y_8 \} \quad (2.27)$$

$$\begin{aligned}
D^2y_{14} = & -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14}) \\
& + \gamma_{32} \left\{ \frac{E_n^* e^{y_9} Dy_9 + P^* e^{y_{11}} Dy_{11} - F^* e^{y_{10}} Dy_{10}}{E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}} + Dy_5 + Dy_{14} - Dy_8 \right\} \\
& + \gamma_{33} \{ \log[E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}} - K_a^* e^{y_{12}} (Dy_{12} + \lambda_1 + \lambda_2)] \\
& - \log[E_n^* + P^* - F^* - K_a^* (\lambda_1 + \lambda_2)] + y_5 + y_{14} - y_8 \}
\end{aligned} \tag{2.28}$$

The equilibrium of the original system (2.1)-(2.14) corresponds to the equilibrium $y_i = 0, i = 1, 2, \dots, 14$ of (2.15)-(2.18). The major advantage of the new system ((2.15)-(2.18)) described by (2.15)-(2.18) is that it is autonomous, but still retains all the dynamic properties of the original system (2.1)-(2.14). In Barnett and He (1999), the paper analyzes the local dynamics of (2.15)-(2.28) in a local neighborhood of the equilibrium, $y_i = 0, i = 1, 2, \dots, 14$. For simplicity, the system (2.15)-(2.28) is denoted as

$$\mathbf{Dx} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}), \tag{2.29}$$

where

$$\mathbf{x} = [y_1 \ Dy_1 \ y_2 \ Dy_2 \ \dots \ y_{14} \ Dy_{14}]' \in R^{28}$$

is the state vector, while

$$\boldsymbol{\theta} = [\beta_1, \dots, \beta_{27}, \gamma_1, \dots, \gamma_{33}, \lambda_1, \lambda_2, \lambda_3]' \in R^{63}$$

is the parameter vector, and $\mathbf{f}(\mathbf{x}, \boldsymbol{\theta})$ is a vector of smooth functions of \mathbf{x} and $\boldsymbol{\theta}$ obtained from (2.15)-(2.28). Note that (2.29) is a first-order system. The point $\mathbf{x}^* = \mathbf{0}$ is an equilibrium of (2.29). Let Θ denote the feasible region determined by those bounds.

2.3. Stability of the Equilibrium

In section 1.2, the discussion on stability describes a means to analyze local stability of the system through linearization. The linearized system of (2.15)-(2.28) is

$$D^2y_1 = -\gamma_1 Dy_1 + \gamma_2 \left\{ \frac{Q^* e^{y_4} + P^* e^{y_{11}}}{Q^* + P^*} - \beta_2 y_7 + (\beta_2 - \beta_3) Dy_5 - y_1 \right\} \tag{2.30}$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left\{ \frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} - y_2 \right\} \quad (2.31)$$

$$D^2 y_3 = -\gamma_5 D y_3 + \gamma_6 \left\{ (1 + \beta_6)(y_4 - y_3) - \frac{y_7 - \beta_7 D y_5}{r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8} \right\} \quad (2.32)$$

$$D^2 y_4 = -\lambda_7 D y_4 + \gamma_8 \left\{ -y_4 - \frac{\beta_9 (q^* p^* / p_i^*)^{\beta_{10}}}{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}}} \beta_{10} (y_5 + y_{14}) \right. \\ \left. + \frac{C^* y_1 + g^* (Q^* y_4 + P^* y_{11}) + K^* D y_3 + K^* (\lambda_1 + \lambda_2) y_3 + E_n^* y_9}{C^* + g^* (Q^* + P^*) + K^* (\lambda_1 + \lambda_2) + E_n^*} \right\} \quad (2.33)$$

$$D^2 y_5 = \gamma_9 (D y_6 - D y_5) + \gamma_{10} \left\{ (1 + \beta_6) \frac{\beta_5 (Q^* / K^*)^{\beta_6}}{1 - \beta_5 (Q^* / K^*)^{\beta_6}} (y_4 - y_3) + y_6 - y_5 \right\} \quad (2.34)$$

$$D^2 y_6 = \gamma_{11} (D y_5 - D y_6) - \gamma_{12} (D y_5 + D y_{14}) + \gamma_{13} \frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} \quad (2.35)$$

$$D^2 y_7 = -\gamma_{14} D y_7 + \gamma_{15} \left\{ -\beta_{14} D y_{14} - y_7 + \frac{\beta_{15}}{M^*} [(Q^* + P^*) p^* (y_5 - y_{13}) + p^* (Q^* y_4 + P^* y_{11})] \right\} \quad (2.36)$$

$$D^2 y_8 = \gamma_{16} (D y_5 + D y_{14} - D y_8) + \gamma_{17} \left\{ (1 + \beta_{10})(y_5 + y_{14}) - y_8 \right. \\ \left. + \frac{C^* y_1 + g^* (Q^* y_4 + P^* y_{11}) + K^* (\lambda_1 + \lambda_2) y_3 + K^* D y_3 + E_n^* y_9}{C^* + g^* (Q^* + P^*) + K^* (\lambda_1 + \lambda_2) + E_n^*} \right\} \quad (2.37)$$

$$D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19} \{ \beta_{18} (y_5 + y_{14}) + y_9 \} \quad (2.38)$$

$$D^2 y_{10} = -[\gamma_{20} + 2(\lambda_1 + \lambda_2)] D y_{10} + \frac{\gamma_{21} \beta_{19}}{F^*} [Q^* (y_4 - y_{10}) + P^* (y_{11} - y_{10})] \quad (2.39)$$

$$D^2 y_{11} = -[\gamma_{22} + 2(\lambda_1 + \lambda_2)] D y_{11} + \gamma_{23} [\beta_{20} + \beta_{21} (r_f^* - \lambda_4)] \frac{K_a^*}{P^*} (y_{12} - y_{11}) \quad (2.40)$$

$$D^2 y_{12} = -[\gamma_{24} + 2(\lambda_1 + \lambda_2)] D y_{12} + \gamma_{25} \left\{ -\beta_{24} \frac{Q^* + P^*}{K_a^*} D y_{14} - \beta_{23} \frac{Q^* + P^*}{K_a^*} y_7 \right. \\ \left. + [\beta_{22} + \beta_{23} (r_f^* - r^*) - \beta_{24} (\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)] \frac{Q^* (y_4 - y_{12}) + P^* (y_{11} - y_{12})}{K_a^*} \right\} \quad (2.41)$$

$$\begin{aligned}
D^2y_{13} &= -\gamma_{26}Dy_{13} - \gamma_{27}y_{13} \\
&+ \gamma_{28} \left\{ \frac{E_n^*Dy_9 + P^*Dy_{11} - F^*Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8 \right\} \\
&+ \gamma_{29} \left\{ \frac{E_n^*y_9 + P^*y_{11} - F^*y_{10} - K_a^*(\lambda_1 + \lambda_2)y_{12} - K_a^*Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8 \right\} \quad (2.42)
\end{aligned}$$

$$\begin{aligned}
D^2y_{14} &= -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14}) \\
&+ \gamma_{32} \left\{ \frac{E_n^*Dy_9 + P^*Dy_{11} - F^*Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8 \right\} \\
&+ \gamma_{33} \left\{ \frac{E_n^*y_9 + P^*y_{11} - F^*y_{10} - K_a^*(\lambda_1 + \lambda_2)y_{12} - K_a^*Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8 \right\} \quad (2.43)
\end{aligned}$$

In matrix form, these equations become

$$\dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}. \quad (2.44)$$

For the set of estimated values of $\{\beta_i\}$, $\{\gamma_j\}$, and $\{\lambda_k\}$ given in Table 2 of Bergstrom, Nowman, and Wymer (1992), all the eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ are stable, having negative real parts, except for the following three:

$$s_1 = 0.0033, \quad s_2 = 0.009 + 0.0453i, \quad s_3 = 0.009 - 0.0453i.$$

Barnett and He (1999) observe that the real parts of the unstable eigenvalues are so small and close to zero, that it is unclear whether they are caused by errors in estimation or the structural properties of the system itself.

Next, they proceed to locate the stable region and the bifurcation boundary by first looking for a stable sub-region of θ and then expanding the sub-region to find its boundary. They first look for a parameter vector $\boldsymbol{\theta}^* \in \theta$ such that (2.44) is stable. They then search for a stable region of $\boldsymbol{\theta}$ and the boundaries of bifurcation regions. To find a $\boldsymbol{\theta}^*$ such that all

eigenvalues of $\mathbf{A}(\boldsymbol{\theta}^*)$ have strictly negative real parts, they first consider the following problem of minimizing the maximum real parts of eigenvalues of matrix $\mathbf{A}(\boldsymbol{\theta})$:

$$\min_{\boldsymbol{\theta} \in \Theta} R_{\max}(\mathbf{A}(\boldsymbol{\theta})) \quad (2.45)$$

where

$$R_{\max}(\mathbf{A}(\boldsymbol{\theta})) = \max_i \{\text{real}(\lambda_i): \lambda_1, \lambda_2, \dots, \lambda_{28} \text{ are eigenvalues of } \mathbf{A}(\boldsymbol{\theta})\}.$$

Barnett and He (1999) could not acquire a closed-form expression for $R_{\max}(\mathbf{A}(\boldsymbol{\theta}))$, since the dimension of $\mathbf{A}(\boldsymbol{\theta})$ is too high for analytic solution. Instead they employ the gradient method to solve the minimization problem (2.45). More precisely, let $\boldsymbol{\theta}^{(0)}$ be the estimated set of parameter values given in Table 2 of Bergstrom, Nowman, and Wymer (1992). At step n , $n \geq 0$, with $\boldsymbol{\theta}^{(n)}$, let

$$\boldsymbol{\theta}^{(n+1)} = \boldsymbol{\theta}^{(n)} - a_n \frac{\partial R_{\max}(\mathbf{A}(\boldsymbol{\theta}))}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{(n)}},$$

where $\{a_n, n = 0, 1, 2, \dots\}$ is a sequence of (positive) step sizes. After several iterations (20 iterations in this case), the algorithm converged to the following point, $\boldsymbol{\theta}^* \in \Theta_1$,

$$\begin{aligned} \boldsymbol{\theta}^* = & [0.9400, 0.2256, 2.3894, 0.2030, 0.2603, 0.1936, 0.1829, 0.0183, 0.2470, \\ & -0.2997, 1.0000, 23.5000, -0.0100, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002, \\ & 0.0097, 0.0049, 0.2812, -0.1000, 44.9030, 0.1431, 0.0004, 71.4241, 0.8213, \\ & 3.9998, 0.8973, 0.6698, 0.0697, 0.1064, 0.0010, 3.9901, 0.3652, 1.0818, \\ & 0.0081, 3.5988, 0.6626, 0.1172, 0.8452, 0.0421, 1.4280, 0.3001, 3.9969, \\ & 3.6512, 3.9995, 4.0000, 3.9995, 3.9410, 0.5861, 0.0040, 0.7684, 0.0427, \\ & 0.1183, 0.0708, 2.3187, 0.1659, 0.0017, 0.0000, 0.0100, 0.0100, 0.0067]. \end{aligned}$$

The corresponding $R_{\max}(\mathbf{A}(\boldsymbol{\theta}^*)) = -0.0039$ implies that all eigenvalues of $\mathbf{A}(\boldsymbol{\theta}^*)$ have strictly negative real parts, and the system (2.44) is locally asymptotically stable around at $\boldsymbol{\theta}^*$.

Barnett and He (1999) then look for the stable region of the parameter space and the bifurcation boundaries starting from this stable point.

2.4. Determination of Bifurcation Boundaries

The goal of this section is to find bifurcation boundaries of the model. Since the linearized system (2.44) only deals with local stability of the system, Barnett and He (1999) deal with local bifurcations as opposed to global bifurcations.

In the previous section, for the set of parameters given in Table 2 of Bergstrom, Nowman, and Wymer (1992), $\mathbf{A}(\boldsymbol{\theta})$ has three eigenvalues with strictly positive real parts. However, at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, found through the gradient method, all eigenvalues of $\mathbf{A}(\boldsymbol{\theta})$ have strictly negative real parts. Since eigenvalues are continuous functions of entries of $\mathbf{A}(\boldsymbol{\theta})$, there must exist at least one eigenvalue of $\mathbf{A}(\boldsymbol{\theta})$ with zero real part on the bifurcation boundary. Different types of bifurcations may occur and three types of bifurcations are discussed in Barnett and He (1999,2002): saddle-node bifurcations, Hopf bifurcations, and transcritical bifurcations.

i. Saddle-node and Hopf Bifurcations

In systems generated by autonomous ordinary differential equations, a saddle-node bifurcation occurs, when the critical equilibrium has a simple zero eigenvalue. If $\det(\mathbf{A}(\boldsymbol{\theta})) = 0$, then $\mathbf{A}(\boldsymbol{\theta})$ has at least one zero eigenvalue. Therefore, Barnett and He (1999) start from $\det(\mathbf{A}(\boldsymbol{\theta})) = 0$ to look for bifurcation boundaries. To demonstrate the feasibility of this approach, Barnett and He (1999) consider the bifurcation boundaries for β_2 and β_5 . The following theorem is proved in Barnett and He (1999) as their theorem 1.

Theorem 2.1. The bifurcation boundary for β_2 and β_5 is determined by

$$1.36\beta_2\beta_5 + 21.78\beta_5 - 2.05\beta_2 - 10.05 = 0. \quad (2.46)$$

A Hopf bifurcation occurs at points at which the system has a nonhyperbolic equilibrium associated with a pair of purely imaginary, but non-zero, eigenvalues and when additional transversality conditions are satisfied. Barnett and He (1999) use the Procedure (P1) introduced in section 1.3.4 to find Hopf bifurcation. They numerically find boundaries of saddle-node bifurcations and Hopf bifurcations for β_2 and β_5 , the surface of the bifurcation boundary for

β_2, β_5 and β_{15} , Hopf bifurcation boundary for γ_8 and β_{15} , and the three dimensional Hopf bifurcation boundary for γ_8, β_{15} and β_2 . Barnett and He (1999) conclude that the method is applicable to any number of parameters.

ii. Transcritical Bifurcations

A new method of finding transcritical bifurcations is proposed in Barnett and He (2002). Again Barnett and He (2002) start from $\det(\mathbf{A}(\boldsymbol{\theta})) = 0$ to look for bifurcation boundaries.

Without loss of generality, Barnett and He (2002) consider bifurcations when two parameters θ_i, θ_j change, while others are kept at $\boldsymbol{\theta}^*$. The matrix $\mathbf{A}(\boldsymbol{\theta})$ is therefore rewritten as

$$\mathbf{A}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta}^*) + \mathbf{B}(\boldsymbol{\theta}^*)\mathbf{D}(\boldsymbol{\mu})\mathbf{C}(\boldsymbol{\theta}^*), \quad (2.47)$$

where $\boldsymbol{\mu} = [\theta_i, \theta_j]$, and $\mathbf{D}(\boldsymbol{\mu})$ is a matrix of appropriate dimension. The dimension of $\mathbf{D}(\boldsymbol{\mu})$ is usually much smaller than that of $\mathbf{A}(\boldsymbol{\theta})$. In this case, the following proposition, proved in Barnett and He (2002) as their Proposition 1, is useful for simplifying the calculation of transcritical bifurcation boundaries.

Proposition 2.1. Assume that $\mathbf{A}(\boldsymbol{\theta})$ has structure (2.47) and that all eigenvalues of $\mathbf{A}(\boldsymbol{\theta}^*)$ have strictly negative real parts. Then $\det(\mathbf{A}(\boldsymbol{\theta})) = 0$, if and only if

$$\det(\mathbf{I} + \mathbf{D}(\boldsymbol{\mu})\mathbf{C}(\boldsymbol{\theta}^*)\mathbf{A}^{-1}(\boldsymbol{\theta}^*)\mathbf{B}(\boldsymbol{\theta}^*)) = 0. \quad (2.48)$$

Barnett and He (2002) demonstrate the usefulness of this approach by considering the bifurcation boundary for $\boldsymbol{\mu} = [\theta_2, \theta_{23}] = [\beta_2, \beta_{23}]$. They find that only the following entries of $\mathbf{A}(\boldsymbol{\theta})$ are functions of $\boldsymbol{\mu}$:

$$\begin{aligned} a_{2,10}(\boldsymbol{\mu}) &= \gamma_2(\beta_2 - \beta_3), & a_{2,13}(\boldsymbol{\mu}) &= -\gamma_2\beta_2, \\ a_{24,7}(\boldsymbol{\mu}) &= \frac{\gamma_{25}\delta Q^*}{K_a^*}, & a_{24,13}(\boldsymbol{\mu}) &= -\frac{\gamma_{25}\beta_{23}(Q^* + P^*)}{K_a^*}, \\ a_{24,21}(\boldsymbol{\mu}) &= \frac{\gamma_{25}\delta P^*}{K_a^*}, & a_{24,23}(\boldsymbol{\mu}) &= -\frac{\gamma_{25}\delta(Q^* + P^*)}{K_a^*} \end{aligned}$$

where $\delta = \beta_{22} + \beta_{23}(r_f - r^*) - \beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)$. In this case, $\mathbf{B}(\boldsymbol{\theta}^*) \in R^{28 \times 2}$ has all zero entries except that its (2,1) entry is 1 and its (24,2) entry is 1. The matrix $\mathbf{C}(\boldsymbol{\theta}^*) \in R^{5 \times 28}$

has zero entries, except the entries are 1 at the following locations: (1,7), (2,10), (3,13), (4,21), (5,23). The matrix $\mathbf{D}(\boldsymbol{\mu})$ is

$$\mathbf{D}(\boldsymbol{\mu}) = \mathbf{d}(\boldsymbol{\mu}) - \mathbf{d}(\boldsymbol{\theta}^*),$$

with

$$\mathbf{d}(\boldsymbol{\mu}) = \begin{bmatrix} 0 & a_{2,10}(\boldsymbol{\mu}) & a_{2,13}(\boldsymbol{\mu}) & 0 & 0 \\ a_{24,7}(\boldsymbol{\mu}) & 0 & a_{24,13}(\boldsymbol{\mu}) & a_{24,21}(\boldsymbol{\mu}) & a_{24,23}(\boldsymbol{\mu}) \end{bmatrix}.$$

Using Proposition 2.1, Barnett and He (2002) observe that $\det(\mathbf{A}) = 0$ is equivalent to

$$\det \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathbf{D}(\boldsymbol{\mu})\mathbf{C}(\boldsymbol{\theta}^*)\mathbf{A}^{-1}(\boldsymbol{\theta}^*)\mathbf{B}(\boldsymbol{\theta}^*) \right) = 0,$$

where

$$\mathbf{C}(\boldsymbol{\theta}^*)\mathbf{A}^{-1}(\boldsymbol{\theta}^*)\mathbf{B}(\boldsymbol{\theta}^*) = \begin{bmatrix} 13.7090 & -17.1187 \\ 0 & 0 \\ -1.7276 & 2.1573 \\ -616.4935 & 389.2039 \\ -616.4935 & 389.2039 \end{bmatrix}.$$

Equivalently, they obtain the bifurcation boundary:

$$-14.23 + 15.91 \theta_2 + 0.28 \theta_{23} - 0.50\theta_2\theta_{23} = 0.$$

When parameters take values on the bifurcation boundary, stability of the system (2.29) needs to be determined by examining the higher order terms in $\mathbf{D}\mathbf{x} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta})$ with center manifold theory. Barnett and He (2002) write $\mathbf{D}\mathbf{x} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta})$ through appropriate coordinate transformation as (see Glendinning (1994) or Guckenheimer and Holmes (1983)):

$$Dx_1 = A_1(\boldsymbol{\theta})x_1 + F_1(x_1, x_2, \boldsymbol{\theta}), \quad (2.49)$$

$$Dx_2 = A_2(\boldsymbol{\theta})x_2 + F_2(x_1, x_2, \boldsymbol{\theta}), \quad (2.50)$$

where all eigenvalues of $A_1(\boldsymbol{\theta})$ have zero real parts and all eigenvalues of $A_2(\boldsymbol{\theta})$ have strictly negative real parts. By center manifold theory, there exists a center manifold, $x_2 = h(x_1)$, such that

$$h(0) = 0 \text{ and } Dh(0) = 0.$$

By substituting $x_2 = h(x_1)$ into (2.49), Barnett and He (2002) obtain

$$Dx_1 = A_1(\boldsymbol{\theta})x_1 + F_1(x_1, h(x_1), \boldsymbol{\theta}). \quad (2.51)$$

The stability of (2.29) is connected to that of (2.51) through the following theorem.

Theorem 2.2. (Henry (1981), Carr (1981)) If the origin of (2.51) is locally asymptotically stable (respectively unstable), then the origin of (2.29) is also locally asymptotically stable (respectively unstable).

By substituting $x_2 = h(x_1)$ into (2.50), Barnett and He (2002) observes that $h(x_1)$ satisfies

$$\begin{aligned} Dx_2 &= Dh(x_1)Dx_1 = Dh(x_1)[A_1(\boldsymbol{\theta})x_1 + F_1(x_1, h(x_1), \boldsymbol{\theta})] \\ &= A_2(\boldsymbol{\theta})h(x_1) + F_2(x_1, h(x_1), \boldsymbol{\theta}), \end{aligned}$$

or $h(x_1)$ satisfies

$$Dh(x_1)[A_1(\boldsymbol{\theta})x_1 + F_1(x_1, h(x_1), \boldsymbol{\theta})] = A_2(\boldsymbol{\theta})h(x_1) + F_2(x_1, h(x_1), \boldsymbol{\theta}), \quad (2.52)$$

$$h(0) = 0, Dh(0) = 0. \quad (2.53)$$

For most cases, especially codimension-1 bifurcations, the dimension of (2.51) is usually one or two. In the case of transcritical bifurcations, the dimension of (2.51) is one. Since solving (2.52) and (2.53) is difficult, Barnett and He (2002) use a Taylor series approximation with several terms to determine the local asymptotic stability or instability of (2.51). In this case, let

$$F_1(x_1, x_2, \boldsymbol{\theta}) = a_1 \frac{x_1^2}{2!} + x_1 a_2 x_2 + a_3 \frac{x_1^3}{3!} + \dots,$$

$$F_2(x_1, x_2, \boldsymbol{\theta}) = b_1 \frac{x_1^2}{2!} + x_1 b_2 x_2 + b_3 \frac{x_1^3}{3!} + \dots.$$

Barnett and He (2002) assume that $h(x_1)$ has the following Taylor expansion

$$h(x_1) = \alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \dots.$$

Then (2.52) becomes

$$\begin{aligned} & \left(ax_1 + \beta \frac{x_1^2}{2!} + \dots \right) \left[A_1(\boldsymbol{\theta})x_1 + a_1 \frac{x_1^2}{2!} + x_1 a_2 \left(\alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \dots \right) + a_3 \frac{x_1^3}{3!} + \dots \right] \\ & = A_2(\boldsymbol{\theta}) \left(\alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \dots \right) + b_1 \frac{x_1^2}{2!} + x_1 b_2 \left(\alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \dots \right) + b_3 \frac{x_1^3}{3!} + \dots . \end{aligned}$$

By comparing coefficients of the same order terms and also observing that $A_1(\boldsymbol{\theta}) = 0$ at a bifurcation point, Barnett and He (1999) observe that

$$\alpha = -A_2^{-1}(\boldsymbol{\theta})b_1, \quad \beta = A_2^{-1}(\boldsymbol{\theta})(\alpha a_1 - b_2 \alpha).$$

Therefore, (2.51) becomes

$$Dx_1 = A_1(\boldsymbol{\theta})x_1 + a_1 \frac{x_1^2}{2!} + \left(\frac{a_2 \alpha}{2!} + \frac{a_3}{3!} \right) x_1^3 + \dots \quad (2.54)$$

The stability analysis of (2.54) determines the stability characteristics of $\mathbf{Dx} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta})$.

Without loss of generality, Barnett and He (2002) consider the stability of the system on the transcritical bifurcation boundary for parameters β_2, β_{23} . Considering the point $(\beta_2, \beta_{23}) = (0.1068, 55.9866)$ on the boundary and using previous approach, Barnett and He (1999) find that (2.51) becomes $Dx_1 = 0.1308 x_1^2 + o(x_1^2)$, which is locally asymptotically unstable at $x_1 = 0$. Therefore, it follows from center manifold theory that the system (2.29) is locally asymptotically unstable at this transcritical bifurcation point. Furthermore, Barnett and He (2002) numerically find boundaries of both Hopf and transcritical bifurcations for θ_2 and θ_{62} , for θ_2, θ_{23} and θ_{62} , for θ_{23} and θ_{62} and for θ_{12}, θ_{23} and θ_{62} .

2.5. Stabilization Policy

We have seen in the previous section that both transcritical and Hopf bifurcations exist in the UK continuous time macroeconomic model. In this section, we provide Barnett and He's (2002) results investigating the control of bifurcations using fiscal feedback laws. They define stabilization policy to be intentional movement of bifurcation regions through policy intervention, with the intent of moving the stable region to include the parameters. However,

there would be no need for stabilization policy, if the parameters were inside the stable region without policy.

Barnett and He (2002) first consider the effect of a heuristically plausible fiscal policy of the following form, as suggested in Bergstrom, Nowman, and Wymer (1992):

$$D \log T_1 = \gamma \left[\beta \log \left\{ \frac{Q}{Q^* e^{(\lambda_1 + \lambda_2)t}} \right\} - \log \left\{ \frac{T_1}{T_1^*} \right\} \right]. \quad (2.55)$$

The control feedback rule (2.55) adjusts the fiscal policy instrument, T_1 , towards a partial equilibrium level, which is an increasing function of the ratio of output to its steady state level. In (2.55), β is a measure of the strength of the feedback, and γ governs the speed of adjustment. According to Bergstrom, Nowman, and Wymer (1992), the control law (2.55) can reduce the positive real parts of unstable eigenvalues through proper choices of parameters β, γ . The intent is for the policy to be stabilizing. However, Barnett and He (2002) tried the following procedure and found that the control law (2.55) is unlikely to stabilize the systems

(2.1)-(2.14). First, they define $y_{15} = \log \left\{ \frac{T_1}{T_1^*} \right\}$, and then they find that y_{15} satisfies

$$Dy_{15} = \gamma \beta y_4 - \gamma y_{15}.$$

They add this equation to the system (2.29) and obtain

$$D\mathbf{w} = \mathbf{A}'(\boldsymbol{\theta})\mathbf{w} + \mathbf{F}'(\mathbf{x}, \boldsymbol{\theta}), \quad (2.56)$$

where

$$\mathbf{w} = \begin{bmatrix} \mathbf{x} \\ y_{15} \end{bmatrix}, \quad \mathbf{F}'(\mathbf{x}, \boldsymbol{\theta}) = \begin{bmatrix} \mathbf{F}(\mathbf{x}, \boldsymbol{\theta}) \\ 0 \end{bmatrix},$$

and $\mathbf{A}'(\boldsymbol{\theta})$ is the corresponding coefficient matrix.

They then consider three sets of parameter values: $\beta = 0.04, \gamma = 0.02$; $\beta = 0.01, \gamma = 0.05$; and $\beta = 0, \gamma = 0$. The case, $\beta = 0, \gamma = 0$, corresponds to the original system (2.1)-(2.14), in which no fiscal policy control is applied. Barnett and He (2002) illustrate the effect of a simple

fiscal policy in three cases, indicating that some stable regions could be destabilized and some unstable regions could be stabilized. But since the feasible region is smaller under control than without control, Barnett and He conclude that the policy is not likely to succeed.

Barnett and He (2002) next consider a more sophisticated fiscal control policy, based upon optimum control theory, with the control being

$$u = \log \left\{ \frac{T_1}{T_1^*} \right\}. \quad (2.57)$$

Under the control (2.57), the system (2.29) becomes

$$\mathbf{D}\mathbf{x} = \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}u + \mathbf{F}(\mathbf{x}, \boldsymbol{\theta}), \quad (2.58)$$

where $\mathbf{B} = [0 \ -\gamma_2 \ 0 \ \dots \ 0]^T \in R^{28}$. The controllability matrix $[\mathbf{B} \ \mathbf{A}\mathbf{B} \ \dots \ \mathbf{A}^{27}\mathbf{B}]$ has rank 7, implying that the pair (\mathbf{A}, \mathbf{B}) is not controllable. Therefore, it is not possible to set the closed-loop eigenvalues of the coefficient matrix of (2.58) arbitrarily.

Nevertheless, the numerical procedure of Khalil (1992) shows that there exists a linear transformation, $\mathbf{z} = \mathbf{T}\mathbf{x}$, such that

$$\mathbf{D}\mathbf{z} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{z} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} u,$$

where $\mathbf{A}_{11} \in R^{21 \times 21}$, $\mathbf{A}_{21} \in R^{7 \times 21}$, $\mathbf{A}_{22} \in R^{7 \times 7}$, $\mathbf{B}_2 = [0 \ \dots \ 0 \ 1] \in R^7$,

$$\mathbf{T}\mathbf{A}(\boldsymbol{\theta})\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{T}\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix},$$

and $(\mathbf{A}_{22}, \mathbf{B}_2)$ is controllable. Further, all eigenvalues of \mathbf{A}_{11} have negative real parts, implying that $(\mathbf{A}(\boldsymbol{\theta}), \mathbf{B})$ is stabilizable.

To obtain a feedback control law stabilizing (2.58), Barnett and He (2002) consider minimizing

$$J = \int_0^{\infty} [\mathbf{x}^T \mathbf{U} \mathbf{x} + V u^2] dt,$$

where $\mathbf{U} \in R^{28 \times 28}$ and $V \in R^1$ are positive definite. According to linear system theory, the optimal feedback control law is given by

$$u = \mathbf{Kx}, \quad \mathbf{K} = -V^{-1}\mathbf{B}^T\mathbf{P},$$

where \mathbf{P} is positive definite and solves the algebraic Riccati equation $\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PB}V^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{U} = \mathbf{0}$.

Choosing $\mathbf{U} = \mathbf{I}$ and $V = 1$, Barnett and He (2002) get

$$\begin{aligned} \mathbf{K} = [& 1.5036, 0.4754, 0.0178, 0.0307, -1.1897, 18.5851, 7.2979, 1.9063, 2.3147, \\ & 23.2392, 0.7488, 7.2091, 38.9965, 39.4000, 0.1841, 0.2129, 0.3061, 0.0494, -0.0027, \\ & 0.0000, -0.0013, -0.0002, 0.9550, 1.8482, -0.3329, -0.5475, 0.9369, -1.0402]. \end{aligned} \quad (2.59)$$

Under the control $u = \mathbf{Kx}$, equation (2.58) becomes

$$\mathbf{Dx} = [\mathbf{A}(\boldsymbol{\theta}) + \mathbf{BK}]x + \mathbf{F}(x, \boldsymbol{\theta}). \quad (2.60)$$

Since all the eigenvalues of $\mathbf{A} + \mathbf{BK}$ have strictly negative real parts under the choice of \mathbf{K} , the state feedback law $u = \mathbf{Kx}$ indeed stabilizes the system (2.60). Barnett and He (2002) also confirm by direct verification that there exist no bifurcations under the control law (2.60) for (β_2, β_5) .

Barnett and He (2002) further investigate whether there is a parameter $\boldsymbol{\theta}' \in \theta$ at which the system (2.60) is unstable. They check the stability of (2.60) under the control law (2.60) for all parameter $\boldsymbol{\theta} \in \theta$. The following $\boldsymbol{\theta}' \in \theta_1$ were found

$$\begin{aligned} \boldsymbol{\theta}' = [& 0.9400, 0.5074, 2.0913, 0.2030, 0.2612, 0.1933, 0.2309, 0.0000, 0.2510, -0.3423, \\ & 1.0000, 23.5000, -0.0100, 0.2086, 0.0332, 13.5460, 0.4562, 0.9322, 0.0100, 0.0034, \\ & 0.1324, -0.5006, 100.0000, 0.0000, 0.0004, 71.4241, 0.8213, 4.0000, 1.0289, 0.3631, \\ & 0.1201, 0.1000, 0.0010, 3.7015, 0.4860, 1.1270, 0.0042, 3.3994, 0.4802, 0.1300, 0.6851, \\ & 0.0620, 1.2134, 0.3830, 4.0000, 3.2535, 3.8592, 4.0000, 4.0000, 3.5723, 0.4775, 0.0071, \end{aligned}$$

0.6104, 0.0143, 0.1718, 0.1227, 2.5551, 0.1833, 0.0035, 0.0000, 0.0018, 0.0004, 0.0100].

The corresponding $R_{\max}(\mathbf{A}(\boldsymbol{\theta}')) = 0.4971$. Hence, there indeed exists a parameter $\boldsymbol{\theta}' \in \theta_1$ at which (2.60) is unstable.

Barnett and He (2002) investigate whether the use of an optimal control feedback policy with a structural model would be easily implemented, if the Lucas critique and time inconsistency issues did not exist. It is often believed that designing such active policy would be easy, if it were not for the problems produced by the Lucas critique and by the time inconsistency of optimal control. However, Barnett and He (2002) find that even without those problems, the design of a successful feedback policy can be difficult. They consider a policy to be successful, if the policy shifts the bifurcation boundaries such that the stable region moves towards the point estimates of the parameters. Then the probability is increased that the stable region will include the values of the parameters. Barnett and He (2002) find that Bergstrom's proposed selection of a fiscal policy feedback rule for his own UK model is counterproductive for three reasons: (1) the resulting policy equation derived from optimal control theory is complicated and depends heavily upon the model; (2) the problem of robustness of the optimal control policy to specification error is not addressed; and (3) the problems of possible time inconsistency of optimal control policy are not taken into consideration. The effects of policy feedback rules can depend upon the complicated geometry of bifurcation boundaries and how they are moved by augmentation of the model by the feedback rule. As a result, Barnett and He (2002) conclude that such policies can be counterproductive.

3. Leeper and Sims Model

3.1. Introduction

Barnett and He (2008) conducted a bifurcation analysis of the best known Euler-equations general-equilibrium macroeconomic model: the Leeper and Sims (1994) model and found the existence of singularity bifurcation boundaries within the parameter space. This section surveys Barnett and He's (2008)'s bifurcation analysis of that model.

Barnett and He (2008) provided initial confirmation of Grandmont's views about bifurcation. Grandmont (1985) found that the parameter space of even the most classical

dynamic general-equilibrium macroeconomic models is stratified into bifurcation regions. This result challenged the prior common view that different kinds of economic dynamics can only be attributed to different kinds of structures. But he was not able to reach conclusions about policy relevance, since his results were based on a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Nevertheless, robustness of dynamical inferences can be seriously damaged by the stratification of a confidence region into bifurcated subsets, when a bifurcation boundary crosses the confidence region of a parameter. Policy relevance was introduced by Barnett and He (1999, 2001a, 2002), who investigated Bergstrom-Wymer continuous-time dynamic macroeconomic model of UK economy. That Keynesian model does permit introduction of welfare improving countercyclical policy. Barnett and Duzhak (2008, 2010) further explored policy relevance by demonstrating the existence of Hopf and flip bifurcations within the more recent class of New Keynesian models.

There is a large literature on dynamic macroeconomic models.⁴ In particular, the Lucas critique has motivated development of Euler-equations models with policy-invariant deep parameters. A seminal example in this class is the Leeper and Sims model, which contains parameters of consumer and firm behavior as deep parameters of tastes and technology. The deep parameters are invariant to government policy rule changes, and hence immune to the Lucas critique.⁵ The dimension of the state space in the Leeper and Sims model is substantially lower than in the Bergstrom--Wymer UK model, but still too high for analysis by available analytical approaches. Through numerical procedures, Barnett and He (2008) find that the dynamics of the Leeper and Sims model are complicated by the model's Euler equations structure. The model consists of both differential equations and algebraic constraints. Barnett and He (2008) found that the order of the dynamics of the Leeper and Sims model could change within a small neighborhood of the estimated parameter values. Within this small neighborhood close to a bifurcation boundary, one eigenvalue of the linearized part of the model can move quickly from finite to infinite and back again to finite. Barnett and He (2008) state that a large stable eigenvalue indicates that some variables can respond rapidly to

⁴ See Barnett and He (2008), footnote 2.

⁵ Similar models are developed in Kim (2000) and in Binder and Pesaran (1999), according to Barnett and He (2008), footnote 3.

changes of other variables. A large unstable eigenvalue indicates one variable's rapid diversion away from other variables, while an infinity eigenvalue indicates existence of a pure algebraic relationships among the variables. Due to the nature of the mapping from parameter space to functional space of dynamical solutions, the sensitivity to the setting of the parameters presents serious challenges to the robustness of dynamical inferences.

Barnett and He's (2008)'s bifurcation analysis of the Leeper and Sims model not only confirm the policy relevance of Grandmont's views but also reveal the existence of a singularity bifurcation boundary within a small neighborhood of the estimated parameter values. Singularity bifurcation, surveyed in section 1, had not previously been encountered in economics, although is known in the engineering and mathematics literatures. On the singularity boundary, the number of differential equations will decrease, while the number of algebraic constraints will increase. Such change in the order of dynamics had not previously been found with macroeconometric models. Barnett and He (2008) speculate that singularity bifurcation may be a common property of Euler equations models. Even though the dimension of the dynamics can be the same on both sides of a singularity bifurcation boundary, the nature of the dynamics on one side may differ dramatically from the nature of the dynamics on the other side. Hence the implications of singularity bifurcation are not limited to the change in the dimension of the dynamics directly on the bifurcation boundary. These results cast into doubt the robustness of dynamical inferences acquired by simulation only at the point estimate of the parameters. Barnett and He (2008) advocate simulating models at various settings throughout the parameters' confidence region, rather than solely at the parameters point estimates.

Since the US data used in the model include imported and exported goods, the Leeper and Sims model, although specified as a closed economy model, is implicitly open economy. Barnett and He (2008) consider extension of their analysis to an explicitly open-economy Euler-equations model. In section 6, we survey research on bifurcation phenomena in explicitly open-economy New Keynesian models.

3.2. The Model⁶

⁶ The model description is modified from Barnett and He (2008).

The Leeper and Sims (1994) model includes the dynamic behavior of consumers, firms, and government. Consumers and firms maximize their respective objective functions, and the government pursues countercyclical policy objectives through monetary and tax policies satisfying an intertemporal government budget constraint. Parameters of consumer and firm behavior are the deep parameters of tastes and technology and are invariant to government policy rule changes. The model consists of both ordinary differential equations and algebraic constraints. The resulting system is called a differential/algebraic system in systems theory. The detailed derivation of the models is available in Leeper and Sims (1994) and will not be repeated in this survey.

The Leeper and Sims model consists of the following 12 state variables.

L = labor supply,

C^* = consumption net of transaction costs,

M = consumer demand for non-interest-bearing money,

D = consumer demand for interest-bearing money,

K = capital,

Y = factor income from capital and labor, excluding interest on government debt,

C = gross consumption,

Z = investment,

X = consumption goods aggregate price,

Q = investment goods price,

V = income velocity of money,

P = general price level.

The consumer maximizes utility according to

$$E \left[\int_0^{\infty} \exp\left(-\int_0^t \beta(s) ds\right) \frac{(C^{*\pi}(1-L)^{1-\pi})^{1-\gamma}}{1-\gamma} dt \right]$$

subject to

$$XC + QZ + \tau + \frac{\dot{M} + \dot{D}}{P} = Y + \frac{iD}{P},$$

$$XC^* + \phi VY = XC,$$

$$\begin{aligned}\dot{K} &= Z - \delta K, \\ Y &= rK + wL + S,\end{aligned}$$

$$V = \frac{PY}{M},$$

where $\pi \in (0,1)$ and $\gamma > 0$ are parameters; $0 \leq \beta(s) \leq 1$ is the subjective rate of time preference at time s ; τ is the level of lump-sum taxes paid by the representative consumer; i is the nominal rate of return earned on government bonds; S is the sum of dividends received by the representative consumer, w is the wage rate; $\varphi > 0$ is the transaction cost per unit of VY ; $\delta \geq 0$ is the rate of depreciation of capital; and $r =$ rental rate of return on capital. Parameters in this model are not necessarily assumed to be constant.

The firms maximize profits according to

$$\max \left\{ X(C + g) + QI^* + A(\alpha K^\sigma + L^\sigma)^{\frac{1}{\sigma}} - rK - wL - ((C + g)^\mu + \theta I^{*\mu})^{\frac{1}{\mu}} \right\},$$

where g is the level of government purchases. The following are parameters:

$$A > 0, \alpha > 0, \theta > 0, \mu \geq 0, \text{ and } 0 \leq \sigma \leq 1.$$

The market-clearing condition is $I^* = Z + nK$, where $n =$ the fraction of existing capital purchased by the government for distribution to the newborn. Investment goods, I^* , produced by the firm include both those bought by the existing population, and those purchased by the government for distribution to the newborn, as indicated by the market-clearing condition.

In this model, the state variables satisfy the following differential equations:

$$\frac{1}{P}(\dot{M} + \dot{D}) = Y - XC - QL + \frac{iD}{P} + \tau \quad (3.1)$$

$$\dot{K} = Z - \delta K \quad (3.2)$$

$$(1 - \pi(1 - \gamma)) \frac{\dot{C}^*}{C^*} + (1 - \gamma)(1 - \pi) \frac{\dot{L}}{1 - L} + \frac{\dot{X}}{X} + \frac{\dot{P}}{P} = i - \beta + \frac{\dot{\pi}}{\pi} + \dot{\pi}(1 - \gamma) \log\left(\frac{C^*}{1 - L}\right) \quad (3.3)$$

$$\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} = i + \delta - (1 - 2\phi V) \frac{r}{Q} \quad (3.4)$$

Equation (3.1) represents the consumers' budget constraint. Equation (3.2) is the law of motion for capital, and equations (3.3) and (3.4) are the first-order conditions derived from the consumers' optimization problem. In addition, the state variables also satisfy the following algebraic constraints.

$$X = \left(\frac{Y}{C + g}\right)^{1-\mu}, \quad (3.5)$$

$$Q = \theta \left(\frac{Y}{Z + nK}\right)^{1-\mu}, \quad (3.6)$$

$$r = A^\sigma \alpha \left(\frac{Y}{K}\right)^{1-\delta}, \quad (3.7)$$

$$w = A^\sigma \left(\frac{Y}{L}\right)^{1-\delta}, \quad (3.8)$$

$$XC^* + \phi VY = XC, \quad (3.9)$$

$$Y = rK + wL + S, \quad (3.10)$$

$$V = \frac{PY}{M}, \quad (3.11)$$

$$X(C + g) + Q(Z + nK) = Y, \quad (3.12)$$

$$(1 - 2\phi V) \frac{w}{X} = \frac{1 - \pi}{\pi} \frac{C^*}{1 - L}, \quad (3.13)$$

$$i = \phi V^2. \quad (3.14)$$

Equations (3.5)-(3.8) are obtained from the first-order conditions of the firms' optimization problem. Equation (3.9) defines consumption net of transaction costs, with total output serving as a measure of the level of transactions at a given point in time. Equation (3.10) defines income. Equation (3.11) is the income velocity of money. Equation (3.12) is the social resources constraint. Equations (3.13)-(3.14) are obtained from the first-order conditions for the consumers' optimization problem.

The control variables consist of the nominal rate of return on government bonds, i , and the level of lump-sum taxes, τ . According to Barnett and He (2008), the monetary policy rule is

$$\frac{\dot{i}}{i} = a_p \log\left(\frac{P}{\bar{P}}\right) + a_{\text{int}} \frac{\dot{P}}{P} + a_i \log\left(\frac{i}{\beta}\right) + a_L \log\left(\frac{L}{\bar{L}}\right) + \varepsilon_i \quad (3.15)$$

and the tax policy is

$$\frac{d}{dt} \frac{\tau}{C} = b_\tau \left(\frac{\tau}{C} - \frac{\bar{\tau}}{C}\right) + b_L \log\left(\frac{L}{\bar{L}}\right) + b_{\text{inf}} \frac{\dot{P}}{P} + b_x \left(\frac{D}{PY} - \frac{\bar{D}}{\bar{P}\bar{Y}}\right) + \varepsilon_\tau \quad (3.16)$$

The free parameters are the steady state debt-to-income level, \bar{D}/\bar{Y} , the steady state price level, \bar{P} , the a 's, and the b 's. The disturbance noises are ε_i and ε_τ . The control variables are i and τ_c . Barnett and He (2008) use $\tau_c = \frac{\tau}{C}$ rather than τ as a control. The exogenous variables are $n, g, \pi, \delta, \theta, \alpha, A$, and ϕ , which are specified by Leeper and Sims to follow logarithmic first-order autoregressive (AR) processes in continuous time, while β is specified to be a logarithmic first-order AR in unlogged form. Barnett and He (2008) analyze the structural properties of (3.1)-(3.14) without external disturbances. Barnett and He (2006b,2008) treat all parameters in (3.3) as fixed parameters and treat the exogenous variables as realized at their measured values. The extension of this analysis to the case of stochastic bifurcation is a subject for future research.

Next Barnett and He (2008) reduce the dimension of the problem by temporarily eliminating some state variables for the convenience of analytical investigation. They contract to the following 7 state variables

$$\mathbf{x} = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \end{bmatrix}. \quad (3.17)$$

The remaining state variables can be written as unique functions of \mathbf{x} . By eliminating M, C^*, V, Q, X from the independent state variables, it can be determined directly from (3.1)-(3.14) that \mathbf{x} satisfies the following equations.

$$\begin{aligned} & \frac{1}{P} \dot{D} + \frac{Y\sqrt{\phi/i}}{P} \dot{P} + (\sqrt{\phi/i}) \dot{Y} \\ & = Y + \frac{iD}{P} - \left(\frac{Y}{C+g}\right)^{1-\mu} C - \theta \left(\frac{Y}{Z+nK}\right)^{1-\mu} L - \tau_c C + \frac{Y\sqrt{i/\phi}}{2V^2\phi} \dot{i} \end{aligned} \quad (3.18)$$

$$\begin{aligned} & (1-\pi(1-\gamma)) \left(\frac{1-\phi V Y^\mu (1-\mu)(C+g)^{-\mu}}{C-\phi V Y^\mu (C+g)^{1-\mu}} - \frac{1-\mu}{C+g} \right) \dot{C} \\ & - \left(\frac{(1-\pi(1-\gamma))\phi V \mu Y^{\mu-1} (C+g)^{1-\mu}}{C-\phi V Y^\mu (C+g)^{1-\mu}} + \frac{1-\mu}{Y} \right) \dot{Y} + \frac{\dot{P}}{P} + \frac{(1-\gamma)(1-\pi)}{1-L} \dot{L} \\ & = i - \beta + \frac{Y^\mu (C+g)^{1-\mu}}{C-\phi V Y^\mu (C+g)^{1-\mu}} \frac{1}{2\sqrt{i\phi}} \dot{i} \end{aligned} \quad (3.19)$$

$$\frac{\dot{P}}{P} + (1-\mu) \left(\frac{\dot{Y}}{Y} - \frac{\dot{Z} + n\dot{K}}{Z+nK} \right) = -(1-2\phi V) \frac{a^\sigma \alpha}{\theta} Y^{\mu-\sigma} (Z+nK)^{1-\mu} K^{\sigma-1} + i + \delta \quad (3.20)$$

$$\dot{K} = Z - \delta K, \quad (3.21)$$

$$0 = (C+g)^\mu + \theta(Z+nK)^\mu - Y^\mu, \quad (3.22)$$

$$0 = \alpha K^\sigma + L^\sigma - a^{-\sigma} Y^{-\sigma} \quad (3.23)$$

$$0 = (1-2\phi V) \frac{a^\sigma Y^{\mu-\sigma} (C+g)^{1-\mu}}{L^{1-\sigma}} + \frac{1-\pi}{\pi} \frac{\phi V}{1-L} Y^\mu (C+g)^{1-\mu} - \frac{1-\pi}{\pi} \frac{C}{1-L} \quad (3.24)$$

Then Barnett and He (2008) write equations (3.18)-(3.24) as

$$\mathbf{h}(\mathbf{x}, \mathbf{u}) \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad (3.25)$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}), \quad (3.26)$$

where \mathbf{x} is a 7-dimensional state vector, \mathbf{u} is a 2-dimensional control vector, $\mathbf{h}(\mathbf{x}, \mathbf{u})$ is a 4×7 -dimensional matrix, and $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is a 4×1 vector of functions, $\mathbf{g}(\mathbf{x}, \mathbf{u})$ is a 3×1 vector of functions. Equation (3.25) describes the nonlinear dynamical behavior of the model, and (3.26) describes the nonlinear algebraic constraints. The system formed by (3.25) and (3.26) is called nonlinear descriptor systems in the mathematical literature. Barnett and He (2006b,2008) use $m = 7, m_1 = 4, m_2 = 3$, and $l = 2$ (with $m = m_1 + m_2$) to denote respectively the dimension of \mathbf{x} , the number of differential equations in (3.25), the number of algebraic constraints in (3.26), and the dimension of the vector of control variables \mathbf{u} .

Barnett and He (2008) solve the steady state of the system (3.25)-(3.26) for the 7 state variables, \mathbf{x} , conditionally on the setting of the controls \mathbf{u} from the following equations:

$$\mathbf{0} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad (3.27)$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{u}). \quad (3.28)$$

and get

$$\begin{aligned} \bar{i} &= \beta \\ \dot{i} &= 0 \end{aligned} \quad (3.29)$$

$$\bar{\tau}_c = \frac{\bar{\tau}}{\bar{C}}$$

The first equation of (3.29) is found from (3.15) in the steady state, the second equation from the definition of steady state, and the third equation from (3.16) in the steady state. The values $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ are solutions to (3.27)-(3.28), and (3.29). The resulting steady state is the equilibrium of (3.25)-(3.26), when the control variables are set at their steady state.

The vector of parameters in the steady state system is

$$\mathbf{p} = [\pi \ \beta \ \theta \ \alpha \ a \ \phi \ \delta \ \mu \ \gamma \ \sigma]'$$

Here g is taken as a fixed value by the private sector at its setting by the government. The constraints on the parameter values and g are:

$$0 < \pi < 1, \gamma > 0, 0 \leq \sigma \leq 1, \mu \geq 1, \delta \geq 0, 0 \leq \beta \leq 1, \delta > 0, g \geq 0. \quad (3.30)$$

3.3. Singularity in Leeper and Sims Model

Barnett and He (2008) use local linearization around the equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ and derive the following linearized system of (3.25) and (3.26):

$$\mathbf{E}_1 \dot{\mathbf{x}} = \mathbf{A}_1 \mathbf{x} + \mathbf{B}_1 \mathbf{u}, \quad (3.31)$$

$$\mathbf{0} = \mathbf{A}_2 \mathbf{x} + \mathbf{B}_2 \mathbf{u}, \quad (3.32)$$

where

$$\mathbf{E}_1 = \mathbf{h}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in R^{m_1 \times m} = R^{4 \times 7}$$

$$\mathbf{A}_1 = \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_1 \times m} = R^{4 \times 7}$$

$$\mathbf{A}_2 = \frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_2 \times m} = R^{3 \times 7}$$

$$\mathbf{B}_1 = \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_1 \times l} = R^{4 \times 2}$$

$$\mathbf{B}_2 = \frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}, \mathbf{u}=\bar{\mathbf{u}}} \in R^{m_2 \times l} = R^{3 \times 2}$$

Barnett and He (2008) find the linearized system satisfies the regularity condition according to Gantmacher (1974). In particular, they find values of the determinant's parameter s such that $\det\left(\begin{bmatrix} s\mathbf{E}_1 - \mathbf{A}_1 \\ -\mathbf{A}_2 \end{bmatrix}\right) \neq 0$. Since the linearized system is regular, it is solvable. Barnett and He (2008) further transform the linearized system (3.31)-(3.32) into the following form.

Definition 3.1 (Barnett and He (2008), Definition 3.1) Two systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (3.33)$$

and

$$\tilde{\mathbf{E}}\dot{\mathbf{y}} = \tilde{\mathbf{A}}\mathbf{y} + \tilde{\mathbf{B}}\mathbf{u} \quad (3.34)$$

are said to be restricted system equivalent (r.s.e), if there exist two nonsingular matrices \mathbf{T}_1 and \mathbf{T}_2 such that

$$\mathbf{T}_1\mathbf{E}\mathbf{T}_2 = \tilde{\mathbf{E}}, \quad \mathbf{T}_1\mathbf{A}\mathbf{T}_2 = \tilde{\mathbf{A}}, \quad \mathbf{T}_1\mathbf{B} = \tilde{\mathbf{B}}, \quad \mathbf{x} = \mathbf{T}_2\mathbf{y}.$$

Barnett and He (2008) note that the form (3.34) can be obtained by using the coordinate transform $\mathbf{x} = \mathbf{T}_2\mathbf{y}$ into (3.33) and then multiplying both sides of (3.33) by \mathbf{T}_1 from the left. They next transformed (3.31)-(3.32) into suitable r.s.e. forms. They denote $r_E = \text{rank}(\mathbf{E}_1)$, where $r_E \in \{1,2,3,4\}$. Then there exist nonsingular matrices \mathbf{T}_1 and \mathbf{T}_2 such that

$$\mathbf{T}_1\mathbf{E}_1\mathbf{T}_2 = \begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

They substitute the form $\mathbf{x} = \mathbf{T}_2 \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$, where $\mathbf{y}_1 \in R^{r_E}$ and $\mathbf{y}_2 \in R^{m-r_E} = R^{7-r_E}$, into (3.31)-(3.32) and also multiply both sides of (3.31) by \mathbf{T}_1 . It follows that (3.31)-(3.32) is r.s.e to

$$\dot{\mathbf{y}}_1 = \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{12}\mathbf{y}_2 + \mathbf{B}_{11}\mathbf{u}, \quad (3.35a)$$

$$\mathbf{0} = \mathbf{A}_{21}\mathbf{y}_1 + \mathbf{A}_{22}\mathbf{y}_2 + \mathbf{B}_{12}\mathbf{u}, \quad (3.35b)$$

$$\mathbf{0} = \mathbf{A}_{31}\mathbf{y}_1 + \mathbf{A}_{32}\mathbf{y}_2 + \mathbf{B}_2\mathbf{u}, \quad (3.35c)$$

where

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{T}_1\mathbf{A}_1\mathbf{T}_2, \quad \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix} = \mathbf{T}_1\mathbf{B}_1, \quad [\mathbf{A}_{31} \quad \mathbf{A}_{32}] = \mathbf{A}_2\mathbf{T}_2,$$

with $\mathbf{A}_{11} \in R^{r_E \times r_E}$, $\mathbf{A}_{12} \in R^{r_E \times (7-r_E)}$, $\mathbf{A}_{21} \in R^{(4-r_E) \times r_E}$, $\mathbf{A}_{22} \in R^{(4-r_E) \times (7-r_E)}$, $\mathbf{A}_{31} \in R^{3 \times r_E}$, $\mathbf{A}_{32} \in R^{3 \times (7-r_E)}$, $\mathbf{B}_{11} \in R^{r_E \times 2}$, and $\mathbf{B}_{12} \in R^{(4-r_E) \times 2}$, while \mathbf{y}_1 is an r_E dimensional vector and \mathbf{y}_2 is a $7 - r_E$ dimensional vector.

Barnett and He (2008) combine equations (3.35a) and (3.35b) and acquire the following:

$$\dot{\mathbf{y}}_1 = \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{12}\mathbf{y}_2 + \mathbf{B}_{11}\mathbf{u}, \quad (3.36a)$$

$$\mathbf{0} = \tilde{\mathbf{A}}_{21}\mathbf{y}_1 + \tilde{\mathbf{A}}_{22}\mathbf{y}_2 + \tilde{\mathbf{B}}_{12}\mathbf{u}, \quad (3.36b)$$

where

$$\tilde{\mathbf{A}}_{21} = \begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{31} \end{bmatrix}, \quad \tilde{\mathbf{A}}_{22} = \begin{bmatrix} \mathbf{A}_{22} \\ \mathbf{A}_{32} \end{bmatrix}, \quad \tilde{\mathbf{B}}_{12} = \begin{bmatrix} \mathbf{B}_{12} \\ \mathbf{B}_2 \end{bmatrix}.$$

If $\tilde{\mathbf{A}}_{22}$ is nonsingular, it follows from (3.36b) that $\mathbf{y}_2 = -(\tilde{\mathbf{A}}_{22})^{-1}(\tilde{\mathbf{A}}_{21}\mathbf{y}_1 + \tilde{\mathbf{B}}_{12}\mathbf{u})$. They substitute the form of \mathbf{y}_2 into (3.36a) and get

$$\dot{\mathbf{y}}_1 = \mathbf{C}\mathbf{y}_1 + \mathbf{D}\mathbf{u}, \quad (3.37)$$

where $\mathbf{C} = \mathbf{A}_{11} - \mathbf{A}_{12}\tilde{\mathbf{A}}_{22}^{-1}\tilde{\mathbf{A}}_{21} \in R^{r_E \times r_E}$ and $\mathbf{D} = \mathbf{B}_{11} - \mathbf{A}_{12}\tilde{\mathbf{A}}_{22}^{-1}\tilde{\mathbf{B}}_{12} \in R^{r_E \times 2}$. This implies that if $\tilde{\mathbf{A}}_{22}$ is nonsingular, given the algebraic relationship between \mathbf{y}_1 and \mathbf{y}_2 in equation (3.36b), the dynamics of \mathbf{y}_1 can be explained in terms of ordinary differential equations (3.37).

Linear system ((3.31), (3.32)) is equivalent to ((3.37), (3.36b)), only when $\tilde{\mathbf{A}}_{22}$ is nonsingular. If $\tilde{\mathbf{A}}_{22}$ were singular, the above transformation would not be possible and singular bifurcation would occur. As explained in Barnett and He (2004,2006b), if $\tilde{\mathbf{A}}_{22}$ becomes exactly singular, the dimension of dynamics change. The dynamics also would change substantially, if $\tilde{\mathbf{A}}_{22}$ moves between two settings located on opposite sides of a singular bifurcation boundary.

To examine the case when $\tilde{\mathbf{A}}_{22}$ is singular in more detail, Barnett and He (2008) rewrite the linearized system ((3.36a), (3.36b)) as

$$\begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} \\ \tilde{\mathbf{B}}_{12} \end{bmatrix} \mathbf{u}. \quad (3.38)$$

The matrix pair $(\begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix})$, which is in the form of a matrix pencil, is also regular, since the model is regular. Therefore, there exist nonsingular matrices, $\tilde{\mathbf{T}}_1$ and $\tilde{\mathbf{T}}_2$, such that (Gantmacher (1974)):

$$\tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{I}_{\tilde{m}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \text{ and } \tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix} \tilde{\mathbf{T}}_2 = \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\tilde{m}_2} \end{bmatrix},$$

where $\tilde{m}_1 + \tilde{m}_2 = m$ and \mathbf{N} is a nilpotent matrix; i.e. there exists a positive integer $d \geq 1$ such that $\mathbf{N}^d = \mathbf{0}$. The smallest such integer d is called the nilpotent index of \mathbf{N} . One example of a nilpotent matrix is:

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (3.39)$$

Barnett and He (2008) next consider the coordinate transform $\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \tilde{\mathbf{T}}_2 \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}$,

substitute it for \mathbf{y} in equation (3.38), and multiply both sides of (3.38) by $\tilde{\mathbf{T}}_1$ from the left. The following r.s.e. form of ((3.31),(3.32)) results:

$$\dot{\mathbf{z}}_1 = \tilde{\mathbf{A}}_1 \mathbf{z}_1 + \tilde{\mathbf{B}}_1 \mathbf{u}, \quad (3.40)$$

$$\mathbf{N} \dot{\mathbf{z}}_2 = \mathbf{z}_2 + \tilde{\mathbf{B}}_2 \mathbf{u}, \quad (3.41)$$

where

$$\begin{bmatrix} \tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{B}}_2 \end{bmatrix} = \tilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix}.$$

The solutions to (3.40) and (3.41) are respectively

$$\mathbf{z}_1 = e^{\tilde{\mathbf{A}}_1(t-t_0)} \mathbf{z}_1(0) + \int_{t_0}^t e^{\tilde{\mathbf{A}}_1(t-\xi)} \tilde{\mathbf{B}}_1 \mathbf{u}(\xi) d\xi,$$

$$\mathbf{z}_2 = -\sum_{k=1}^{d-1} \delta^{(k-1)}(t) \mathbf{N}^k \mathbf{z}_2(0) - \sum_{k=0}^{d-1} \mathbf{N}^k \tilde{\mathbf{B}}_2 \mathbf{u}^{(k)}(t),$$

where $t_0 \geq 0$ is the initial time, $\delta^{(k-1)}(t)$ is the derivative of order $k - 1$ of the Dirac delta function, and $\mathbf{u}^{(k)}$ denotes that k -th order derivative of \mathbf{u} .

If $\mathbf{N} = \mathbf{0}$, it follows from (3.41) that $\mathbf{z}_2 = -\tilde{\mathbf{B}}_2 \mathbf{u}$, which is a smooth algebraic relationship between \mathbf{z}_2 and \mathbf{u} ; and the above solution for \mathbf{z}_2 does not apply. Only when \mathbf{N} is nonzero, there exist impulsive terms involving the Dirac delta functions, which could produce shock effects in the first summation of the solution for \mathbf{z}_2 , and smooth derivative terms of \mathbf{u} in the second

summation. The solution structure with nonzero \mathbf{N} is very different from the solution of ordinary differential equations as in (3.40) for \mathbf{z}_1 .

The following theorem links bifurcation phenomena at $\mathbf{N} \neq \mathbf{0}$ to the singularity of $\tilde{\mathbf{A}}_{22}$. The proof is contained in Barnett and He (2008), Theorem 3.1.

Theorem 3.1. If both (3.40)-(3.41) and (3.36a)-(3.36b) are r.s.e forms of the same linearized system (3.31)-(3.32), then $\mathbf{N} = \mathbf{0}$, if and only if $\tilde{\mathbf{A}}_{22}$ is nonsingular. Hence it follows that

$$\det(\tilde{\mathbf{A}}_{22}) \neq 0.$$

The next theorem links the singularity of $\tilde{\mathbf{A}}_{22}$ to the rank of the original coefficient matrix. The proof is contained in Barnett and He (2008), Theorem 3.2.

Theorem 3.2. Assume that \mathbf{E}_1 has full row rank, i.e.

$$\text{rank}(\mathbf{E}_1) = m_1.$$

Then $\tilde{\mathbf{A}}_{22}$ is nonsingular, if and only if the $m \times m$ matrix $\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}$ is nonsingular, so that

$$\text{rank} \left(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix} \right) = m.$$

Theorem 3.2 provides the condition for the existence of a singularity bifurcation boundary, so that $\det \left(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix} \right) = 0$.

The following corollary says that the singularity condition does not change whenever state variables that can be modeled by ordinary differential equations are added or deleted. The proof is contained in Barnett and He (2008), Corollary 3.1.

Corollary 3.1. Consider the following system describing the dynamics of (\mathbf{x}, \mathbf{v}) , where $\mathbf{v} \in R^{m_3}$ for arbitrary m_3 .

$$\mathbf{E}_1 \dot{\mathbf{x}} + \mathbf{E}_{1v} \dot{\mathbf{v}} = \mathbf{A}_1 \mathbf{x} + \mathbf{A}_{1v} \mathbf{v} + \mathbf{B}_1 \mathbf{u}, \quad (3.42a)$$

$$\dot{\mathbf{v}} = \mathbf{A}_v \mathbf{v} + \mathbf{B}_v \mathbf{u}, \quad (3.42b)$$

$$\mathbf{0} = \mathbf{A}_2\mathbf{x} + \mathbf{A}_{2v}\mathbf{v} + \mathbf{B}_2\mathbf{u}, \quad (3.42c)$$

where $\mathbf{E}_{1v}, \mathbf{A}_{1v}, \mathbf{A}_v, \mathbf{B}_v, \mathbf{A}_{2v}$ are arbitrary matrices of dimension

$$m_1 \times m_3, m_1 \times m_3, m_3 \times m_3, m_3 \times l, \text{ and } m_2 \times m_3,$$

respectively, and the other matrices are as defined above. Then the singularity condition for (3.42a), (3.42b), and (3.42c) is the same as that for ((3.31), (3.32)).

The above corollary says that adding (or deleting) state variable that can be modeled by ordinary differential equations does not change the singularity condition. The corollary is useful in reducing the dimension of the problem under consideration. With this corollary, Barnett and He (2008) are able to drop the Leeper and Sims' model's state variable K from the state vector (3.17) in the system ((3.31), (3.32)) without affecting the singularity condition. The singularity condition then becomes

$$\det \left(\begin{bmatrix} \mathbf{E}'_1 \\ \mathbf{A}'_2 \end{bmatrix} \right) = 0, \quad (3.43)$$

in which

$$\mathbf{E}'_1 = \begin{bmatrix} \frac{1}{p} & \frac{Y}{PV} & 0 & 0 & 0 & \frac{1}{V} \\ 0 & \frac{1}{P} & e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & 0 & e_{26} \\ 0 & \frac{1}{P} & 0 & 0 & -\frac{1-\mu}{Z+nK} & \frac{1-\mu}{Y} \end{bmatrix}$$

and

$$\mathbf{A}'_2 = \begin{bmatrix} 0 & 0 & \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & \mu Y^{\mu-1} \\ 0 & 0 & a_{23} & a_{24} & 0 & a_{26} \\ 0 & 0 & 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma} \sigma Y^{\sigma-1} \end{bmatrix}$$

with

$$e_{23} = \frac{1-\pi(1-\gamma)}{C^*} [1 - \phi V Y^\mu (\mu-1)(C+g)^{\mu-2}] - \frac{1-\mu}{C+g},$$

$$e_{26} = \frac{1 - \pi(1 - \gamma)}{C^*} [-\phi V Y^\mu \mu (C + g)^{\mu-1}] + \frac{1 - \mu}{Y},$$

$$a_{23} = (1 - 2\phi V) A^\sigma Y^{\mu-\sigma} L^{\sigma-1} (1 - \mu) (C + g)^{-\mu} - \frac{1 - \pi}{\pi} \frac{1}{1 - L},$$

$$a_{24} = (1 - 2\phi V) A^\sigma Y^{\mu-\sigma} (\sigma - 1) L^{\sigma-2} (C + g)^{1-\mu} - \frac{1 - \pi}{\pi} \frac{C}{(1 - L)^2},$$

$$a_{26} = (1 - 2\phi V) A^\sigma (\mu - \sigma) Y^{\mu-\sigma-1} L^{\sigma-1} (C + g)^{1-\mu}.$$

The prime denotes the deletion of the state variable K from the vector \mathbf{x} in equation (3.17) and deletion of equation (3.21), which is the corresponding differential equation for capital K .

Barnett and He (2008) also show by direct calculation that (3.43) is equivalent to

$$\det \begin{pmatrix} e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & \frac{1-\mu}{Z+nK} & e'_{26} \\ \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & -\mu Y^{\mu-1} \\ a_{23} & a_{24} & 0 & a_{26} \\ 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma} \sigma Y^{\sigma-1} \end{pmatrix} = 0 \quad (3.44)$$

where

$$e'_{26} = \frac{1 - \pi(1 - \gamma)}{C^*} [-\phi V Y^\mu \mu (C + g)^{\mu-1}].$$

Equation (3.44) determines the singularity-induced bifurcation boundary. According to Barnett and He (2008), this is the first time that this type of bifurcation has been found in a macroeconomic model.

To investigate bifurcation of the closed-loop system under the control of the monetary policy rule and tax policy rule introduced in (3.15) and (3.16), Barnett and He (2008) augment the state variable to include two more controls as follows:

$$\mathbf{x}_c = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \\ i \\ \tau_c \end{bmatrix}. \quad (3.45)$$

The corresponding linearized system (3.31)-(3.32) becomes

$$\mathbf{E}_1^c \dot{\mathbf{x}}_c = \mathbf{A}_1^c \mathbf{x}_c, \quad (3.46)$$

$$\mathbf{0} = [\mathbf{A}_2 \quad \mathbf{0}] \mathbf{x}_c, \quad (3.47)$$

where $\mathbf{E}_1^c \in R^{m_1^c \times m^c} = R^{6 \times 9}$, $\mathbf{A}_1^c \in R^{m_1^c \times m^c} = R^{6 \times 9}$, $m_1^c = m_1 + 2$, $m^c = m + 2$.

3.4. Numerical Results

Corollary 3.1 allows adding (or deleting) state variables that can be modeled by ordinary differential equations without changing the singularity condition. Barnett and He (2008) then apply condition (3.44) to the closed-loop system (3.47) and look for bifurcation boundaries. They vary pairs of parameters with all other parameters set at their estimates. They also find the intersection of their theoretically feasible ranges and the 95% confidence intervals of their estimated values, in particular, the intersection \mathcal{E} of (3.30) and $[\bar{p}(i) - \bar{c}\sigma_i, \bar{p}(i) + \bar{c}\sigma_i]$, where $\bar{p}(i)$ is the estimated value of parameter $p(i)$, σ_i is the standard error of the estimate, and \bar{c} is the critical value of the 95th-percentile confidence interval for $N(0,1)$.

The estimation information for the parameters μ , g , and β is taken directly from the Leeper and Sims paper, which is presented in Table 3.1⁷.

⁷ Table 3.1 is a replicate of Barnett and He (2008), Table 1.

Table 3.1. Estimation of μ , g , and β

Parameter	Estimate	Standard Error	\mathcal{E} Interval
μ	1.0248	0.324	[1, 1.6598]
g	0.0773	0.292	[0, 0.6496]
β	0.1645	0.288	[0, 0.7290]

Note: Since g is an exogenous variable, rather than a parameter, the “estimate” is the sample mean, and the “standard error” is the sample standard deviation.

Barnett and He (2008) display a few representative sections of the singularity bifurcation boundary. One section is μ versus g , the other is μ versus β . They then explore what happens when β crosses the singularity boundary, with β ranging between 0.08 and 0.24. Table 3.2 displays the changes of finite eigenvalues, $\lambda_1, \dots, \lambda_8$, corresponding to the changes of β .⁸

Table 3.2. Eigenvalue Changes

β	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_1	1.002	1.002	1.002	1.002	1.002	1.002	1.002
λ_2	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_3	-0.303	-0.262	-0.220	-0.215	-0.210	-0.178	-0.135
λ_4	-3.558	-3.559	-3.561	-3.561	-3.561	-3.563	-3.566
λ_5	-0.098	-0.084	-0.077	-0.076	-0.075	-0.072	-0.069
λ_6	-0.002	-0.003	-0.003	-0.003	-0.003	-0.004	-0.004
λ_7	3.101	5.177	8.237	8.682	9.254	13.416	28.401
λ_8	-117.790	-204.703	-1811.413	∞	1456.294	195.888	58.059

Three more infinite eigenvalues are not shown in Table 3.2. The second through the ninth rows are the corresponding finite eigenvalues of the linearized model at each setting of β

⁸ Table 3.2 is a replicate of Barnett and He (2008), Table 2.

shown in the first row. Table 3.2 shows that when the value of β crosses the bifurcation boundary, with β ranging between 0.08 and 0.24, λ_8 decreases from negative values rapidly to $-\infty$, jumps suddenly from $-\infty$ to $+\infty$, and then decreases while remaining positive. This phenomenon shows that the model has a change in dynamic structure, when β crosses the singularity-induced bifurcation boundary. The two regions separated by the boundary exhibit drastically different dynamical behaviors. Barnett and He (2008) also display that very small changes in μ can cause bifurcation independently of the setting of g or β . They also state that the number of dynamic equations and the number of algebraic equations change, when the singularity-induced bifurcation boundary is reached.

4. New Keynesian Model⁹

4.1. Introduction

This section surveys Barnett and Duzhak's (2008, 2010) work on bifurcation analysis within the class of New Keynesian models. Their interest in exploring bifurcation in New Keynesian models is driven by the increasing policy interest in New Keynesian models. In Barnett and Duzhak (2008, 2010), they have studied different specifications of monetary policy rules within the New Keynesian functional structure and have found both the existence of Hopf bifurcation and the existence of period doubling (flip) bifurcation boundaries through numerical procedures.

The usual New Keynesian log-linearized model consists of a forward-looking IS-curve describing consumption smoothing behavior, a Phillips curve derived from price optimization by monopolistically competitive firms in the presence of nominal rigidities, and a monetary policy rule having different specifications. Barnett and Duzhak (2010) use eigenvalues of the linearized system to locate Hopf bifurcation boundaries and investigate different monetary policy effects on bifurcation boundary locations for each case. They use two types of New Keynesian models: one can be reduced to produce a 2×2 Jacobian, and the other produces a 3×3 Jacobian. In the 3×3 case, Barnett and Duzhak (2010) employ a theorem on Hopf bifurcation from the engineering literature.

⁹ This section is summarized from Barnett and Duzhak (2008,2010).

Starting from Grandmont's findings with a classical model, Barnett and Duzhak (2008, 2010) continue to follow the path from the Bergstrom-Wymer UK model, then to the Euler equations Leeper and Sims' macroeconomic models, and then to New Keynesian models. Barnett and Duzhak (2008, 2010) believe that Grandmont's conclusions appear to hold for all categories of dynamic macroeconomic models and suggest that Barnett and He's initial findings with the Bergstrom-Wymer 's UK model appear to be generic. Barnett and Duzhak (2008, 2010) suggest that study of the full nonlinear system and analysis of continuous-time New Keynesian models will merit future research.

4.2. The Model¹⁰

The main assumption of New Keynesian economic theory is that there are nominal price rigidities preventing prices from adjusting immediately and thereby creating disequilibrium unemployment. Price stickiness is often introduced in the manner proposed by Calvo (1983). The model used by Barnett and Duzhak (2008, 2010) is based upon Walsh (2003), section 5.4.1, pp. 232-239, which in turn is based upon the monopolistic competition model of Dixit and Stiglitz (1977).

The model consists of consumers, firms, and monetary policy authority. The representative consumer can allocate wealth to money and bonds and choose the aggregate consumption stream to maximize the utility. Consumers derive utility from the composite consumption good C_t , real money balances, and leisure, and supply labor in a competitive labor market, while receiving labor income $w_t N_t$. Consumers own the firms, which produce consumption goods, and they receive all profits π_t .

Firms operate in a monopolistically competitive market, in which each firm has pricing power over the goods it sells. A random fraction of firms does not adjust its product price in each period. A result is price rigidity by the firm, while the remaining firms adjust prices to their optimal levels. Firms make their production and price-setting decisions by solving the cost minimization and pricing decision problems, such that

¹⁰ The model description is modified from Barnett and Duzhak (2010).

$$x_t = E_t x_{t+1} - \frac{i_t - E_t \pi_{t+1}}{\sigma} \quad (4.1)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \quad (4.2)$$

where π_t is the inflation rate at time t ; i_t is the interest rate; $x_t = (\hat{y}_t - \hat{y}_t^f)$ is the gap between actual output percentage deviation \hat{y}_t and the flexible-price output percentage deviation \hat{y}_t^f ; σ is a degree of relative risk aversion; E_t is the expectations operator, conditionally upon information at time t , and β is the discount factor.

Equation (4.1) represents the demand side of the economy and is a forward-looking IS curve that relates the output gap to the real interest rate. Equation (4.2) represents the supply side and is the New-Keynesian Phillips curve describing how inflation is driven by the output gap and expected inflation. The remaining equation to close the model will be a monetary policy rule, in which the central bank uses a nominal interest rate as the policy instrument. Two main policy classes are targeting rules and instrument rules. A well-known instrument rule is Taylor's rule, using a reaction function responding to inflation and output to set the path of the Federal Funds rate. Barnett and Duzhak (2010) initially center analysis on specification of the current-looking Taylor rule, then on forward-looking, backward-looking, and hybrid Taylor rules. Literature also proposes many ways to define an inflation target. Barnett and Duzhak (2010) consider current-looking, forward-looking and backward-looking inflation targeting policies.

4.3. Determinacy and Stability Analysis

Barnett and Duzhak (2010) use Theorem 1.1 for the analysis of the reduced 2×2 case of $\mathbf{A}E_t \mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t$. They also find bifurcations in the 3×3 case by using the following Lemma 4.1 and Theorem 4.1, which arise from the engineering literature. That approach had not previously been used in the economics literature. According to Barnett and Duzhak (2010), in the 3×3 case with current-looking or backward-looking policy rules, the only form of bifurcation detected from the linearized model was Hopf bifurcation.

Lemma 4.1. (Barnett and Duzhak (2010), Lemma 3.1) For a matrix $\mathbf{A} = [a_{ij}]$, with $i, j = 1, 2, 3$, a pair of complex conjugate eigenvalues lies on the unit circle and another eigenvalue lies inside the unit circle, if and only if

- (a) $|x| < 1$,
- (b) $|x + z| < 1 + y$,
- (c) $y - xz = 1 - x^2$,

where z, y , and x are the coefficients of the characteristic equation $\lambda^3 + z\lambda^2 + y\lambda + x = 0$ of the matrix \mathbf{A} .

The following theorem is Barnett and Duzhak's (2010), Theorem 3.2. The proof is included in that paper.

Theorem 4.1. (Existence of Hopf Bifurcation in 3 Dimensions) Consider a map $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \boldsymbol{\varphi})$, where \mathbf{x} has 3 dimensions. Let \mathbf{J} be the Jacobian of the transformation, and let the characteristic polynomial of the Jacobian be $P(\lambda) = \lambda^3 + z\lambda^2 + y\lambda + x = 0$. Assume that for one of the equilibria, $(\mathbf{x}^*, \boldsymbol{\varphi}^*)$, there is a critical value, φ_i^c , for one of the parameters, φ_i^* , in $\boldsymbol{\varphi}^*$ such that eigenvalue conditions (a),(b), and (c) and transversality condition (d) hold, where:

- (a) $|x| < 1$,
- (b) $|x + z| < 1 + y$,
- (c) $y - xz = 1 - x^2$,
- (d) $\frac{\partial |\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*} \Big|_{\varphi_i^* = \varphi_i^c} \neq 0$ for the complex conjugates with $j = 1, 2$.

Then there is an invariant closed curve Hopf-bifurcating from $\boldsymbol{\varphi}^*$.

i. Current-Looking Taylor Rule

The current-looking Taylor rule is:

$$i_t = a_1 \pi_t + a_2 x_t, \tag{4.3}$$

where a_1 is the coefficient of the central bank's reaction to inflation and a_2 is the coefficient of the central bank's reaction to the output gap.

The 3-equation system ((4.1), (4.2),(4.3)) constitutes a New Keynesian model. To analyze the model's determinacy and stability properties, Barnett and Duzhak (2010) first display the system in the following form, which is not a closed form:

$$\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t + \boldsymbol{\delta}_t,$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & \frac{1}{\sigma} \\ -\kappa & 1 & 0 \\ a_2 & a_1 & -1 \end{bmatrix}, \mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix}.$$

Obtaining the matrix $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$ is impossible, since A is a singular matrix.

Therefore, they reduce the system to a system of two log-linearized equations by substituting Taylor's rule (4.3) into the consumption Euler equation. The system of two equations has the following form:

$$\begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

which can be written as

$$\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t,$$

where

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\ -\kappa & 1 \end{bmatrix}.$$

Premultiply the system by the inverse matrix \mathbf{A}^{-1} ,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -\frac{1}{\beta\sigma} \\ 0 & \frac{1}{\beta} \end{bmatrix},$$

results in

$$E_t x_{t+1} = \mathbf{C}x_t \quad \text{or} \quad \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{a_2\beta + \kappa}{\sigma\beta} & \frac{a_1\beta - 1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$.

The eigenvalues of \mathbf{C} are the roots of the characteristic polynomial

$$p(\lambda) = \det(\mathbf{C} - \lambda\mathbf{I}) = \lambda^2 - \lambda \left[1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta} \right] + \frac{\sigma\beta + a_2\beta + \kappa a_1\beta}{\sigma\beta^2}.$$

Define D as

$$D = \left[1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta} \right]^2 - 4 \frac{\sigma\beta + a_2\beta + \kappa a_1\beta}{\sigma\beta^2}.$$

Then the eigenvalues are

$$\lambda_1 = \frac{1}{2} \left(1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta} + \sqrt{D} \right) \quad \text{and} \quad \lambda_2 = \frac{1}{2} \left(1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta} - \sqrt{D} \right).$$

According to Blanchard and Kahn (1980), the system of expected difference equations has a determinate solution, if the number of eigenvalues outside the unit circle equals the number of forward looking variables. This system has two forward-looking variables, x_{t+1} and π_{t+1} . Therefore the stability and uniqueness of the solution require both eigenvalues to be outside the unit circle. It can be shown that both eigenvalues will be outside the unit circle, if and only if

$$(a_1 - 1)\kappa + (1 - \beta)a_2 > 0. \tag{4.4}$$

Interest rate rules that satisfy $a_1 > 1$ are called active. Such active rules define Taylor's principle, stating that the interest rate should be set higher than the increase in inflation. When $a_1 > 1$, clearly (4.4) holds. Monetary policy satisfying the Taylor's principle is thought to eliminate equilibrium multiplicities.

In this case, the Jacobian of the New Keynesian model can be written in the form:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{a_2\beta + \kappa}{\sigma\beta} & \frac{a_1\beta - 1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

The model is parameterized by:

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \end{pmatrix}$$

Barnett and Duzhak (2008, 2010) use a_1 and a_2 as candidates for bifurcation parameters. They employ Theorem 1.1. to look for the existence of Hopf bifurcation for this New Keynesian model with current looking Taylor rule. The following result is proved in Barnett and Duzhak's (2008), Proposition 3.1:

Proposition 4.1. The new Keynesian model with current-looking Taylor rule, equations (4.1),(4.2) and (4.3), undergoes a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_2^c = \sigma\beta - \kappa a_1 - \sigma$.

Based on the result in Prop. 4.1, Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{\sigma + \sigma\beta - \kappa a_1\beta + \kappa}{\sigma\beta^2} < 1.$$

ii. Forward-Looking Taylor Rule

A forward-looking Taylor rule is:

$$i_t = a_1 E_t \pi_{t+1} + a_2 E_t x_{t+1}. \quad (4.5)$$

The model consisting of (4.1), (4.2) and (4.5) is parameterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \end{pmatrix}.$$

The resulting Jacobian has the following form:

$$\mathbf{J} = \begin{bmatrix} \frac{\sigma}{\sigma - a_2} + \frac{\kappa(1 - a_1)}{(\sigma - a_2)\beta} & \frac{a_1 - 1}{(\sigma - a_2)\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

Barnett and Duzhak (2010) use a_1 and a_2 as candidates for bifurcation parameters. The following result is proved in Barnett and Duzhak (2008) as Proposition 3.2:

Proposition 4.2. The new Keynesian model with forward-looking Taylor rule, equations (4.1), (4.2) and (4.5), undergoes a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_2^c = -\frac{\sigma}{\beta} + \sigma$.

Based on the result in Prop. 4.2, Barnett and Duzhak (2010) find the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{1}{2} \left(\beta + \frac{\kappa(1 - a_1)}{\sigma} + \frac{1}{\beta} \right) < 1.$$

Barnett and Duzhak (2010) propose a numerical algorithm to detect a period doubling bifurcation, which is based on the following technique. Given the i^{th} iterate of the fixed point, $f^i(\mathbf{x}) - \mathbf{x} = 0$, a period-doubling bifurcation will occur whenever $\varphi_{PD} = 0$ with $\varphi_{PD} = \det(\mathbf{J}^{(i)} + \mathbf{I}_n)$, where $\mathbf{J}^{(i)}$ is the Jacobian matrix of the iterated map f^i .

Barnett and Duzhak (2010) use the software continuation package CONTENT, developed by Yuri Kuznetsov and V.V. Levitin, to locate the bifurcation boundary. Barnett and Duzhak select the parameter a_2 to be a free bifurcation parameter and find a period-doubling bifurcation point at $a_2 = 2.994$, with the other parameters set constant in accordance with their paper's appendix table. The nature of the state space solution depends upon where the

bifurcation boundary is located. If parameter a_2 is moved to 3 with the other parameters set constant, the solution becomes periodic. Along the bifurcation boundary, the values of parameter, a_2 , are between 2.75 and 3. When values of a_1 and a_2 are along the bifurcation boundary with the forward looking Taylor rule, Barnett and Duzhak (2010) find that the central bank actively reacts to the expected future values of inflation and even more aggressively to the forecasted values of the output gap.

iii. Hybrid Taylor Rule

A Hybrid-Taylor rule is:

$$i_t = a_1 E_t \pi_{t+1} + a_2 x_t \quad (4.6)$$

This rule was proposed in Clarida, Gali, and Gertler (2000), who maintain that the rule reflects the Federal Reserve's existing policy.

The system ((4.1),(4.2),(4.6)) has the following Jacobian:

$$J = \begin{bmatrix} 1 + \frac{a_2}{\sigma} + \frac{\kappa(1-a_1)}{\sigma\beta} & \frac{a_1-1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

Barnett and Duzhak (2010) use a_1 and a_2 as candidates for bifurcation parameters. The following result was proved in Barnett and Duzhak (2008), Proposition 3.3:

Proposition 4.3. The new Keynesian model with Hybrid-Taylor rule, equations, (4.1),(4.2), and (4.6), undergoes a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_2^c = \beta\sigma - \sigma$.

Based on Proposition 4.3, Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{\sigma(1 + \beta^2) + \kappa(1 - a_1)}{2\sigma\beta} < 1.$$

iv. Current-Looking Inflation Targeting

The inflation targeting equation is:

$$i_t = a_1 \pi_t, \quad (4.7)$$

which can be used instead of the Taylor rule to complete the New Keynesian model.

The system ((4.1),(4.2),(4.7)) has the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} \frac{\sigma\beta + \kappa}{\sigma\beta} & \frac{a_1\beta - 1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

The model is characterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \end{pmatrix}.$$

Barnett and Duzhak (2010) use a_1 as a candidate for a bifurcation parameter. The following result is proved in Barnett and Duzhak (2008), Proposition 3.4:

Proposition 4.4. The new Keynesian model with current-looking inflation targeting, equations (4.1),(4.2) and (4.7), produces a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $a_1^c = \frac{\sigma\beta - \sigma}{\kappa}$.

Based on Proposition 4.4, Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\sigma + \kappa}{\sigma\beta} < 1.$$

v. Forward-Looking Inflation Targeting

A forward-looking inflation targeting rule is:

$$i_t = a_1 E_t \pi_{t+1}. \quad (4.8)$$

The system ((4.1),(4.2),(4.8)) has the Jacobian as follows:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{\kappa(1-a_1)}{\sigma\beta} & \frac{a_1-1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

The model is parameterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \end{pmatrix}.$$

The following proposition is proved in Barnett and Duzhak (2008), Proposition 3.5:

Proposition 4.5. The new Keynesian model with forward-looking inflation targeting, equations (4.1),(4.2), and (4.8), produces a Hopf bifurcation at equilibrium points, if and only if the discriminant of the characteristic equation is negative and $\beta^c = 1$.

Based on Proposition 4.5, which does not depend on a_1 , Barnett and Duzhak (2010) find that the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\kappa(a_1 - 1)}{2\sigma} < 1.$$

Parameter β is both the discount factor and the coefficient in (4.2) which scales the impact of expected inflation. Assuming for simplicity that $\beta = 1$, Barnett and Duzhak (2010) find it surprising that this common setting of parameter β can put the model directly onto a Hopf bifurcation boundary. This conclusion is conditional upon the assumption that the model is a good approximation to the economy and that the discriminant of the characteristic equation is negative. In such cases, it is not appropriate to set $\beta = 1$.

Barnett and Duzhak (2010) further find that the dynamic solution in phase space, i.e. with inflation rate plotted against output gap, will be periodic, if $\beta = 0.98$. They find that if the parameter value is located directly on the bifurcation boundary, solution in phase space will become an invariant limit cycle.

vi. Backward-Looking Taylor Rule

Backward-looking monetary policy rules are intended to prevent expectations driven fluctuations. Such rules are constructed with decisions based on observed past values of variables. Examples are found in Carlstrom and Fuerst (2000) and Eusepi (2005). Barnett and Duzhak (2010) observe that such a policy should be sufficient for determinacy of equilibria.

In a backward-looking Taylor rule, the central bank sets an interest rate according to the past values of inflation and output gap as follows:

$$i_t = a_1\pi_{t-1} + a_2x_{t-1}. \quad (4.9)$$

The system ((4.1),(4.2),(4.9)) can be written in the following form:

$$E_t \mathbf{x}_{t+1} = \mathbf{C} \mathbf{x}_t,$$

with

$$\mathbf{C} = \begin{bmatrix} 1 + \frac{\kappa}{\sigma\beta} & -\frac{1}{\sigma\beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ a_2 & a_1 & 0 \end{bmatrix}, \quad \mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix}.$$

Matrix \mathbf{C} has the characteristic polynomial

$$p(\lambda) = \det(\mathbf{C} - \lambda\mathbf{I}) = \lambda^3 - \frac{\sigma(1 + \beta) + \kappa}{\sigma\beta} \lambda^2 + \frac{\sigma - \beta a_2}{\sigma\beta} \lambda + \frac{\kappa a_1 + a_2}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010), Proposition 3.6.

Proposition 4.6. The New Keynesian model with backward-looking Taylor rule produces a Hopf

bifurcation at equilibrium points, if the transversality condition $\frac{\partial |\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*} \Big|_{\varphi_i^* = \varphi_i^c} \neq 0$ holds,

and if the parameters α_1 and α_2 satisfy the following three conditions at the equilibrium:

$$(a) \quad \left| \frac{a_2 + \kappa a_1}{\sigma\beta} \right| < 1,$$

$$(b) a_2(1 - \beta) + \kappa(a_1 - 1) > 0,$$

$$(c) \frac{\sigma - \beta a_2}{\sigma\beta} + \frac{(\kappa a_1 + a_2)(\sigma(1 + \beta) + \kappa)}{\sigma^2 \beta^2} = 1 - \left(\frac{\kappa a_1 + a_2}{\sigma\beta}\right)^2.$$

vii. Backward-Looking Inflation Targeting

A backward-looking inflation targeting rule sets the interest rate according to inflation during a previous period, as follows:

$$i_t = a_1 \pi_{t-1}. \quad (4.10)$$

The system ((4.1),(4.2),(4.10)) has the Jacobian as follows:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{\kappa}{\sigma\beta} & -\frac{1}{\sigma\beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ 0 & a_1 & 0 \end{bmatrix}.$$

The Jacobian has the characteristic polynomial

$$p(\lambda) = \lambda^3 - \frac{\sigma(1 + \beta) + \kappa}{\sigma\beta} \lambda^2 + \frac{1}{\beta} \lambda + \frac{\kappa a_1}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010) as Proposition 3.7.

Proposition 4.7. The New Keynesian model with backward-looking inflation targeting produces

a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial |\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*} \Big|_{\varphi_i^* = \varphi_i^c} \neq 0$

holds, and if the parameters φ_i^* satisfy the following three conditions at the equilibrium:

$$(a) \left| \frac{\kappa a_1}{\sigma\beta} \right| < 1,$$

$$(b) a_1 > 1,$$

$$(c) \frac{\sigma^2 \beta + \kappa a_1 (\sigma(1 + \beta) + \kappa)}{\sigma^2 \beta^2} = 1 - \left(\frac{\kappa a_1}{\sigma \beta} \right)^2.$$

Barnett and Duzhak (2010) note that their numerical search for bifurcations in this class of models has found only Hopf bifurcations.

viii. Current-Looking Taylor Rule with Interest Rate Smoothing Term

A current-looking Taylor rule with interest rate smoothing term allows central bankers to avoid volatility in interest rate by including a lagged interest rate term in the rule as follows:

$$i_t = (1 - a_3)(a_1 \pi_t + a_2 x_t) + a_3 i_{t-1}. \quad (4.11)$$

Parameter a_3 , which is assumed to be between 0 and 1, describes the degree of interest rate smoothing by the central bank. The model consisting of (4.1), (4.2) and (4.11) is parameterized by

$$\boldsymbol{\varphi} = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

The model has the following matrix form:

$$E_t \mathbf{x}_{t+1} = \mathbf{C} \mathbf{x}_t,$$

with

$$\mathbf{C} = \begin{bmatrix} 1 + \frac{\kappa}{\sigma \beta} & -\frac{1}{\sigma \beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ -a_2(a_3 - 1) + \frac{(-1 + a_3)(a_1 \sigma - a_2) \kappa}{\sigma \beta} & -\frac{(-1 + a_3)(a_1 \sigma - a_2)}{\sigma \beta} & -\frac{a_2(-1 + a_3)}{\sigma} + a_3 \end{bmatrix}$$

and

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix}.$$

This system has the following characteristic polynomial:

$$p(\lambda) = \lambda^3 + \left(\frac{a_2(a_3-1)}{\sigma} - 1 - a_3 - \frac{\kappa}{\sigma\beta} - \frac{1}{\beta} \right) \lambda^2 + \left(\frac{\kappa a_1 - a_2 a_3 + a_2 + \kappa a_3(1-a_1)}{\sigma\beta} + a_3 + \frac{1+a_3}{\beta} \right) \lambda - \frac{a_3}{\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010), Proposition 3.8.

Proposition 4.8. The New Keynesian model consisting of ((4.1),(4.2) ,(4.11)) produces a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial |\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*} \Big|_{\varphi_i^* = \varphi_i^c} \neq 0$ holds,

and if the parameters, $\boldsymbol{\varphi}^*$, satisfy the following three conditions at the equilibrium:

(a) $a_3 - \beta < 0,$

(b) $a_1 > 1,$

(c) $\frac{1-a_3^2}{\beta} - (1-a_3) + \frac{a_3(a_3-1)}{\beta^2} + \frac{a_3 a_2(a_3-2) + \kappa a_1(1-a_3) + a_2 + \kappa a_3}{\sigma\beta} = 0.$

ix. Backward-Looking Taylor Rule With Interest Rate Smoothing Term

The backward-looking Taylor rule with interest rate smoothing is:

$$i_t = (1 - a_3)(a_1 \pi_{t-1} + a_2 x_{t-1}) + a_3 i_{t-1}. \quad (4.12)$$

The model consisting of (4.1), (4.2) and (4.12) has the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{\kappa}{\sigma\beta} & -\frac{1}{\sigma\beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ a_2(1-a_3) & a_1(1-a_3) & a_3 \end{bmatrix}$$

with characteristic polynomial

$$p(\lambda) = \lambda^3 - \left(1 + a_3 + \frac{\kappa}{\sigma\beta} + \frac{1}{\beta}\right)\lambda^2 + \left(\frac{a_2\beta(a_3 - 1) + \kappa a_3 + \sigma(1 + a_3)}{\sigma\beta} + a_3\right)\lambda + \frac{\kappa a_1(1-a_3) + a_2(1-a_3) - \sigma a_3}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010) as Proposition 3.9.

Proposition 4.9. The New Keynesian model consisting of ((4.1), (4.2), (4.12)) produces a Hopf

bifurcation at equilibrium points, if the transversality condition $\frac{\partial |\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*} \Big|_{\varphi_i^* = \varphi_i^c} \neq 0$ holds,

and if the parameters, $\boldsymbol{\varphi}^*$, satisfy the following three conditions at the equilibrium:

$$(a) \quad \left| \frac{\kappa a_1(1-a_3) + a_2(1-a_3) - \sigma a_3}{\sigma\beta} \right| < 1,$$

$$(b) \quad \left| \frac{\kappa a_1(1-a_3) + a_2(1-a_3) - \sigma a_3 - \kappa - \sigma}{\sigma\beta} - 1 - a_3 \right| < \frac{a_2\beta(a_3 - 1) + \kappa a_3 + \sigma(1 + a_3)}{\sigma\beta} + a_3,$$

$$(c) \quad a_3 + \frac{a_2\beta(a_3 - 1) + \kappa a_3 + \sigma(1 + a_3)}{\sigma\beta} + \frac{((a_2 + \kappa a_1)(1-a_3) - \sigma a_3)(\sigma\beta(1 + a_3) + \kappa + \sigma)}{(\sigma\beta)^2}$$

$$= 1 - \left(\frac{(a_2 + \kappa a_1)(1-a_3) - \sigma a_3}{\sigma\beta} \right)^2$$

Through numerical procedures, Barnett and Duzhak (2010) also find the existence of period-doubling bifurcation by varying a_2 , while holding other parameters fixed in accordance with the appendix in Barnett and Duzhak (2010). The first period doubling bifurcation point is found at $a_2 = 5.7$. Starting from this point, Barnett and Duzhak (2010) then vary a_2 and a_3

simultaneously. They discover that period doubling bifurcation will occur for large values of the parameter a_2 . As a result, aggressive reaction of the central bank to past values of the output gap can lead to a period doubling bifurcation within this model.

Duzhak (2010) started from point $a_2 = 5.7$ and varied parameters a_2 and a_1 simultaneously, while holding the other parameters constant in accordance with their paper's appendix. They numerically found a period doubling bifurcation boundary with values of the parameter a_2 within a very narrow range from 5.98 to 6.02. Barnett and Duzhak (2010) also found that a change in the interest rate smoothing parameter a_3 leads to a different critical period-doubling bifurcation value for the parameter a_2 . Although previously thought to be the least prone to any kind of bifurcations, backward-looking interest rate rules show evidence of both Hopf bifurcation and period-doubling bifurcation.

x. Hybrid Rule With Interest Rate Smoothing Term

The hybrid rule with interest rate smoothing, proposed in Clarida, Gali and Gertler (1998), is often believed to match the empirics of Japan, the United States, and the European Union. That rule allows the central banker to set a short-term interest rate based on forecasted inflation, the current value of the output gap, and a past value of the interest rate, as follows:

$$i_t = (1 - a_3)(a_1\pi_{t+1} + a_2x_t) + a_3i_{t-1}. \quad (4.13)$$

The model consisting of equation ((4.1), (4.2) , (4.13)) can be written as

$$\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t,$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} & 0 \\ 0 & \beta & 0 \\ 0 & -a_1(1 - a_3) & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & \frac{1}{\sigma} \\ -\kappa & 1 & 0 \\ a_2(1 - a_3) & 0 & a_3 \end{bmatrix}, \quad \mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \\ i_{t-1} \end{bmatrix}.$$

This model has the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{\kappa}{\sigma\beta} & -\frac{1}{\sigma\beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ \frac{a_1(1-a_3)\kappa}{\beta} + a_2(1-a_3) & \frac{a_1(1-a_3)}{\beta} & a_3 \end{bmatrix}$$

with characteristic polynomial

$$p(\lambda) = \lambda^3 - \left(1 + a_3 + \frac{\kappa}{\sigma\beta} + \frac{1}{\beta}\right)\lambda^2 + \left(a_3 + \frac{1+a_3}{\beta} - \frac{a_2(1-a_3)}{\sigma} + \frac{a_3\kappa + a_1\kappa(1-a_3)}{\sigma\beta}\right)\lambda - \frac{a_3}{\beta} - \frac{a_3a_2}{\sigma\beta} + \frac{a_2}{\sigma\beta}.$$

The following proposition is proved in Barnett and Duzhak (2010), Proposition 3.10.

Proposition 4.10. The New Keynesian model consisting of ((4.1), (4.2),(4.13)) produces a Hopf bifurcation at equilibrium points, if the transversality condition $\frac{\partial |\lambda_j(\mathbf{x}^*, \boldsymbol{\varphi}^*)|}{\partial \varphi_i^*} \Big|_{\varphi_i^* = \varphi_i^c} \neq 0$ holds, and if the parameters, $\boldsymbol{\varphi}^*$, satisfy the following three conditions at the equilibrium:

$$(a) \quad \left| -\frac{a_3}{\beta} - \frac{a_3a_2}{\sigma\beta} + \frac{a_2}{\sigma\beta} \right| < 1,$$

$$(b) \quad \left| \frac{a_2(1-a_3) - \kappa}{\sigma\beta} - 1 - a_3 - \frac{1}{\beta} - \frac{a_3}{\beta} \right| < 1 + a_3 + \frac{1+a_3}{\beta} - \frac{a_2(1-a_3)}{\sigma} + \frac{a_3\kappa + a_1\kappa(1-a_3)}{\sigma\beta},$$

$$(c) \quad a_3 + \frac{1+a_3}{\beta} - \frac{a_2(1-a_3)}{\sigma} + \frac{a_3\kappa + a_1\kappa(1-a_3)}{\sigma\beta}$$

$$+ \left(-\frac{a_3}{\beta} + \frac{a_2(1-a_3)}{\sigma\beta} \right) \left(1 + a_3 + \frac{1}{\beta} + \frac{\kappa}{\sigma\beta} \right) = 1 - \left(-\frac{a_3}{\beta} + \frac{a_2(1-a_3)}{\sigma\beta} \right)^2.$$

Through numerical procedures, Barnett and Duzhak (2010) find the existence of period-doubling bifurcation by varying a_2 while holding other parameters fixed in accordance with

their appendix. The critical value of parameter a_2 is found at $a_2 = 3.03$. Starting with this point, Barnett and Duzhak (2010) first vary parameters a_2 and a_3 and then vary parameters a_2 and a_1 with the other parameters held constant.

In the first case, they find a fold flip bifurcation point at $a_2 = 3.03$ and $a_2 = 0.46$. In the second case, they find parameter a_2 is located mostly between 3 and 3.15 within the period-doubling bifurcation boundary, regardless of the values of parameter a_1 . They conclude that a period doubling bifurcation will occur, if the central bank actively reacts to the output gap. Therefore, two types of bifurcations are revealed for the hybrid interest rate rule.

5. New Keynesian Model With Regime Switching¹¹

5.1. Introduction

Monetary policy has seen major changes over the past decades. In the 1970s, the central bank stayed relatively passive in its actions in the presence of high inflation along with slow economic growth. Afterwards to help to combat high inflation present at the start of the 1980s, the Federal Reserve shifted to a more active regime. The phenomenon “great moderation” arose from the following period of moderate inflation along with stable economic growth in the mid-1980s. In the 21st century, following the financial crises starting in 2007, the Fed had to move aggressively.

Section 5, based on Barnett and Duzhak (2014), investigates whether bifurcations can result from monetary policy regime switching over time. Barnett and Duzhak (2014) focus on New Keynesian models. Previous literature like Gali and Gertler (1999), Bernanke, Laubach, Mishkin, and Posen (1999), and Leeper and Sims (1994) has shown that the original New Keynesian model has been developed into an important tool for monetary policy. In Barnett and Duzhak (2008) and Barnett and Duzhak (2010), the parameter space of the standard New Keynesian model has been shown to be stratified into bifurcation subsets. Relevant previous work includes, but is not limited to the following. Andrews (1993) and Evans (1985) study

¹¹ This section is summarized from Barnett and Duzhak (2014).

monetary policy with parameter instability. Davig and Leeper (2006) and Farmer, Waggoner, and Zha (2007) study determinacy when the Taylor rule is generalized to allow for regime switching. There is a literature on methods to determine parameter instability in time series (see Hansen (1992) and Nyblom (1989)). Economic models of regime switching had been investigated previously in different contexts, such as Hamilton (1989) and Warne (2000). Clarida, Gali, and Gertler (1999), Sims and Zha (2006), and Groen and Mumtaz (2008) find empirical support for regime switching in monetary policy.¹²

In Barnett and Duzhak (2014), the policy regime is assumed to follow a Markov chain with a fixed transition matrix. As a result, the solution to the model evolved differently depending on the state of the regime. Barnett and Duzhak (2014) investigate three models—a basic setup with a simple monetary policy rule, a New Keynesian model with regime switching, and a New Keynesian model with a hybrid monetary policy rule. They show through bifurcation analysis that regime switching can bring changes in the qualitative properties of the solution.

In the first model, the nominal interest rate is set as a function of current inflation with the response coefficient depending on the policy regime present at the time. Combining both the Fisher equation that links the nominal interest rate to future inflation, and the equation of real interest rate, Barnett and Duzhak (2014) get an equation that relates future inflation to current inflation and the real interest rate. A system of two linear difference equations is acquired for inflation in the two regimes. Barnett and Duzhak(2014) further use the eigenvalues of the system's matrix to perform the bifurcation analysis. Two main findings with respect to bifurcations are: first, for the basic setup, Barnett and Duzhak (2014) find no possibility of a Hopf bifurcation; second, they find the existence of a period doubling bifurcation. In this case, the solution can move from a stable to a periodic solution, where periodicity doubles in successive bifurcations.

In the second model, Barnett and Duzhak (2014) explore whether their analysis of this simple setup carries over to the standard New Keynesian model with regime switching and a standard Taylor rule. The Taylor (1999) rule makes the nominal interest rate a function of both

¹² This model description is modified from Barnett and Duzhak (2014)

inflation and the output gap. Barnett and Duzhak (2014) use numerical methods and find that this model does not exhibit any bifurcations for the range of feasible parameter combinations.

In the third model, Barnett and Duzhak (2014) investigate whether a state-of-the-art hybrid Taylor rule exhibits bifurcations. In this model, the Taylor rule allows for forward looking response to inflation. Using the same technique, they find that this model might exhibit a period-doubling bifurcation. The ideas from the basic setup thus carry over to the more prominent model of monetary policy. The analysis reveals that period doubling bifurcations and the resulting changes in the dynamics in inflation and output have more tendencies to arise in models with the forward-looking Taylor rule than in the model with the current-looking counterpart.

5.2. Dynamics with a Simple Monetary Policy Rule

The basic setup with simple monetary policy rule consists of the following two equations:

$$i_t = \alpha(s_t)\pi_t, \quad (5.1)$$

$$i_t = E_t\pi_{t+1} + r_t. \quad (5.2)$$

A policy reacts to inflation by changing an interest rate according to (5.1),

where i_t is the nominal interest rate, $\alpha(s_t)$ a state-dependent coefficient which changes with the policy regime s_t , and π_t denotes the rate of inflation.

Under the assumption that there are two possible realizations for the policy regime, s_t , the linear reaction function to inflation evolves stochastically between two states, $s_t = 1$ and $s_t = 2$, so that

$$\alpha(s_t) = \begin{cases} \alpha_1 & \text{for } s_t = 1 \\ \alpha_2 & \text{for } s_t = 2, \end{cases}$$

where α_i denotes different parameters that govern the aggressiveness of policy to combat inflation. An active policy regime is the one with policy parameter $\alpha_i > 1$. In Barnett and Duzhak (2014), the active regime is regime 1. The policy regime evolves according to a Markov

chain, where the transitional probabilities are given by the transition matrix with entries $p_{ij} = P[s_t = j | s_{t-1} = i]$ where $i, j = 1, 2$.

Following Davig and Leeper (2006), Barnett and Duzhak (2014) use the Fisher equation (5.2) as the second equation in the model, where r_t is the real interest rate. The Fisher equation links the nominal interest rate to expected inflation and the real interest rate. Barnett and Duzhak (2014) use this relationship to solve for expected inflation, which evolves as a function of the nominal and real interest rates.

Combining (5.1) and (5.2), Barnett and Duzhak (2014) acquire the following dynamic system:

$$\begin{bmatrix} E_t[\pi_{1,t+1}] \\ E_t[\pi_{2,t+1}] \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} r_t \\ r_t \end{bmatrix}.$$

In this model, the real interest r_t is exogenously given. A fully specified macroeconomic model endogenizes this rate.

As is standard in the (bifurcation) analysis of difference equations, Barnett and Duzhak study the economy with parameter certainty. Parameter certainty in that model means that agents have no uncertainty about the level of inflation, if a certain state occurs. This does not mean agents know the level of inflation in the following period: the state of the policy regime determines inflation, and the state of the policy regime itself switches with given probabilities. Using parameter certainty, Barnett and Duzhak (2014) restate the system of linear difference equations as

$$\begin{bmatrix} \pi_{1,t+1} \\ \pi_{2,t+1} \end{bmatrix} = \begin{bmatrix} \frac{p_{22}\alpha_1}{p_{11}p_{22} - p_{12}p_{21}} & \frac{-p_{12}\alpha_2}{p_{11}p_{22} - p_{12}p_{21}} \\ \frac{-p_{21}\alpha_1}{p_{11}p_{22} - p_{12}p_{21}} & \frac{p_{11}\alpha_2}{p_{11}p_{22} - p_{12}p_{21}} \end{bmatrix} \begin{bmatrix} \pi_{1,t} \\ \pi_{2,t} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} r_t \\ r_t \end{bmatrix}.$$

Since the entries in the transition matrix are probabilities, it follows that $p_{11} + p_{21} = 1$ and $p_{22} + p_{12} = 1$. Hence, $\Delta = p_{11}p_{22} - p_{12}p_{21}$ as $= p_{11} + p_{22} - 1$.

To analyze the stability of the evolution of inflation and its dynamic properties, as shown by the linear system above, Barnett and Duzhak (2014) first consider the Jacobian matrix and corresponding characteristic polynomial of the above linear system:

$$\mathbf{J} = \begin{bmatrix} \frac{p_{22}\alpha_1}{p_{11}+p_{22}-1} & \frac{-p_{12}\alpha_2}{p_{11}+p_{22}-1} \\ \frac{-p_{21}\alpha_1}{p_{11}+p_{22}-1} & \frac{p_{11}\alpha_2}{p_{11}+p_{22}-1} \end{bmatrix}$$

$$P(\lambda) = \lambda^2 - b\lambda + c \text{ with } b = \frac{p_{22}\alpha_1 + p_{11}\alpha_2}{p_{11} + p_{22} - 1} \text{ and } c = \frac{\alpha_1\alpha_2}{p_{11} + p_{22} - 1}.$$

The determinant D of the Jacobian matrix is given by

$$D = \left[\frac{p_{22}\alpha_1 + p_{11}\alpha_2}{p_{11} + p_{22} - 1} \right]^2 - \frac{4\alpha_1\alpha_2}{p_{11} + p_{22} - 1}.$$

For a Hopf bifurcation to exist, the discriminant D must be negative, giving a rise to complex roots of $P(\lambda)$. Given that $(p_{11} + p_{22} - 1)^2$ is always nonnegative, it follows that $D < 0$, which is equivalent to $(p_{22}\alpha_1 + p_{11}\alpha_2)^2 - (p_{11} + p_{22} - 1)4\alpha_1\alpha_2 < 0$. The term on the left-hand side stays positive within the feasible set of parameters. Therefore, a Hopf bifurcation which arises only when the roots are complex, is not possible for this economy.

Barnett and Duzhak (2014) further examine the possibility of a period doubling bifurcation. Lemma 1 in Barnett and Duzhak (2014, page 10) provide conditions for the existence of the period doubling bifurcation (see Kuznetsov (1998), p.415). Both conditions for the period doubling bifurcation hold in this model. According to Barnett and Duzhak (2014), if one of the roots of the characteristic polynomial is in the negative part of the unit circle, there is a possibility of a period doubling bifurcation. They then analyze the eigenvalues of the characteristic polynomial. The characteristic polynomial $P(\lambda)$ has the following roots:

$$\lambda_{1,2} = \frac{1}{2} \left[\frac{\alpha_1 p_{22} + \alpha_2 p_{11}}{p_{11} + p_{22} - 1} \pm \sqrt{D} \right]$$

where D is the discriminant defined above.

According to Lemma 1 in Barnett and Duzhak (2014), they need one of the roots to be equal to -1. Setting $\lambda_{1,2} = -1$, the condition becomes

$$\sqrt{(p_{22}\alpha_1 + p_{11}\alpha_2)^2 - (p_{11} + p_{22} - 1)4\alpha_1\alpha_2} = 2(p_{11} + p_{22} - 1) + (p_{22}\alpha_1 + p_{11}\alpha_2),$$

which needs to hold for a period doubling bifurcation to occur. The above expression is simplified as

$$p_{11}(1 + \alpha_2) + p_{22}(1 + \alpha_1) + \alpha_1\alpha_2 = 1. \quad (5.3)$$

Equation (5.3) is a bifurcation boundary, in the form of a function of the parameters of the dynamical model.

To calibrate the economy, Barnett and Duzhak (2014) use the values in Table 5.1¹³. One of the policy regimes, regime 1, is active with a coefficient greater than 1, whereas regime 2 is a passive regime. They further assume that $p_{11} = 0$ is zero, which is the probability of remaining in the active regime, conditional on being in the active regime. Whenever regime 1 occurs, the economy will be sent to a passive regime with certainty.

¹³ Table 5.1. is a replicate of Barnett and Duzhak's (2014) Table 1.

Table 5.1. Standard Parameter Combinations

Parameter	Value
α_1	1.5
α_2	0.5
γ_1	0.3
γ_2	0.15
p_{11}	0.85
p_{22}	0.9
β	0.98
κ	0.024
σ	0.3

Using these assumptions, Barnett and Duzhak (2014) find the critical value for the transitional probability p_{22} to be $p_{22}^c = 0.1$. They use this point as a benchmark to trace out the bifurcation boundary. Varying the other parameters, i.e. policy parameters α_1 and α_2 , along with the probability of staying in the passive regime p_{22} , Barnett and Duzhak (2014) demonstrate a period doubling bifurcation boundary as a function of the three control parameters p_{22} , α_1 , and α_2 . If $p_{22} = 1$, then the policy regime would be passive and stay passive indefinitely. In this case, $1 + \alpha_1 + \alpha_1\alpha_2 = 1$, so no bifurcation can arise. If $p_{22} = 0$, then $\alpha_1\alpha_2 = 1$. The bifurcation boundary is symmetric with respect to the policy parameters α_1 and α_2 . If the policy reaction coefficient α_2 of the passive regime is small, the policy response coefficient α_1 needs to be very large for a bifurcation to arise.

5.3. New Keynesian Model with Regime Switching

The standard New Keynesian model, as laid out in, e.g., Woodford (2003) or Walsh (2003), traditionally consists of the following equations:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) + u_t^D \quad (5.4)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t^S. \quad (5.5)$$

$$i_t = \alpha(s_t)\pi_t + \gamma(s_t)x_t \quad (5.6)$$

Equation (5.4) is the forward-looking IS equation describing the demand side of the economy, and equation (5.5) is the Phillips curve representing the supply side. The IS curve (5.4) relates the output gap, x_t , to the nominal interest rate, i_t , and expectations about the future output gap as well as inflation. The coefficient $\frac{1}{\sigma}$ is the inverse of relative risk aversion, which equals the elasticity of intertemporal substitution, since preferences with constant relative risk aversion are assumed in deriving the equations. The New Keynesian Phillips curve, (5.5), describes how inflation is driven by the output gap and expected inflation. Both equations for demand and supply side allow for a shock, u_t . A rule for monetary policy is (5.6), which takes the form described in Taylor (1999). According to that Taylor rule, the monetary authority sets the nominal interest rate by targeting both inflation and the output gap, where α_i governs the Central bank's reaction to inflation and γ_i the reaction to the output gap.

The model can be written in matrix notation

$$\mathbf{A}\mathbf{Y}_{t+1} = \mathbf{B}\mathbf{Y}_t + \mathbf{u}_t, \quad (5.7)$$

where \mathbf{Y} denotes the vector of variables $\mathbf{Y} = [\pi_{1t} \ \pi_{2t} \ x_{1t} \ x_{2t}]^T$ and \mathbf{u}_t the vector of aggregate demand and supply shocks, while \mathbf{A} and \mathbf{B} are given by

$$\mathbf{A} = \begin{bmatrix} \beta p_{11} & \beta(1-p_{22}) & 0 & 0 \\ \beta(1-p_{11}) & \beta p_{22} & 0 & 0 \\ \frac{p_{11}}{\sigma} & \frac{1-p_{22}}{\sigma} & p_{11} & 1-p_{22} \\ \frac{1-p_{11}}{\sigma} & \frac{p_{22}}{\sigma} & 1-p_{11} & p_{22} \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -\kappa & 0 \\ 0 & 1 & 0 & -\kappa \\ \frac{\alpha_1}{\sigma} & 0 & 1 + \frac{\gamma_1}{\sigma} & 0 \\ 0 & \frac{\alpha_2}{\sigma} & 0 & 1 + \frac{\gamma_2}{\sigma} \end{bmatrix}.$$

Rearranging (5.7), Barnett and Duzhak (2014) obtain the normal form

$$\mathbf{Y}_{t+1} = \mathbf{C}\mathbf{Y}_t + \mathbf{A}^{-1}\mathbf{u}_t, \quad (5.8)$$

where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$.

Now the system is 4-dimensional instead of having a two-by-two Jacobian matrix in the basic form. Since the 4-dimensional model is more difficult to analyze, Barnett and Duzhak (2014) employ the software continuation package CONTENT developed by Yuri Kuznetsov and V.V.Levitin to trace out bifurcation boundaries. Barnett and Duzhak (2014) hold constant the parameters that describe the probabilities of regime, while varying structural and policy parameters. They find that neither a Hopf nor a periodic doubling bifurcation can occur for any feasible set of parameters. They do find a bifurcation for parameter values $\gamma_2 = 0.179$ and $\kappa = -0.46$. However, negative values for κ are economically nonfeasible. In this case, the bifurcation boundary never crosses into the subspace of feasible parameter combinations.

5.4. New Keynesian Model with a Hybrid Monetary Policy Rule

Barnett and Duzhak (2014) further provide an analysis of a state-of-the-art model of a monetary policy. Proposed by Clarida, Gali and Gertler (1999), the model consists of a hybrid rule, which includes both a current-looking and a forward-looking component:

$$i_t = \alpha(s_t)\pi_{t+1} + \gamma(s_t)x_t. \quad (5.9)$$

According to the rule, a policy maker is forward-looking with respect to inflation and current looking with respect to the output gap. The corresponding linear system is:

$$\mathbf{Y}_{t+1} = \mathbf{D}\mathbf{Y}_t,$$

where matrix \mathbf{D} is given by

D =

$$\left[\begin{array}{cccc} \frac{p_{22}}{\beta(-1+p_{22}+p_{11})} & \frac{-1+p_{22}}{\beta(-1+p_{22}+p_{11})} & \frac{-p_{22}k}{\beta(-1+p_{22}+p_{11})} & -\frac{(-1+p_{22})k}{\beta(-1+p_{22}+p_{11})} \\ \frac{-1+p_{11}}{\beta(-1+p_{22}+p_{11})} & \frac{p_{11}}{\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{11})k}{\beta(-1+p_{22}+p_{11})} & -\frac{p_{11}k}{\beta(-1+p_{22}+p_{11})} \\ \frac{p_{22}(-1+\alpha_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(-1+\alpha_2)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{p_{22}(k-k\alpha_1+\sigma\beta+\beta\gamma_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(k-k\alpha_2+\sigma\beta+\beta\gamma_2)}{\sigma\beta(-1+p_{22}+p_{11})} \\ \frac{(-1+p_{11})(-1+\alpha_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+\alpha_2)p_{11}}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{11})(k-k\alpha_1+\sigma\beta+\beta\gamma_1)}{\sigma\beta(-1+p_{22}+p_{11})} & \frac{p_{11}(k-k\alpha_1+\sigma\beta+\beta\gamma_2)}{\sigma\beta(-1+p_{22}+p_{11})} \end{array} \right]$$

Numerical analysis of this dynamic system to find Hopf and period doubling bifurcations leads to two findings, which are the same as for the simple economy. First, there is no possibility of a Hopf bifurcation. Second, a period doubling bifurcation emerges.

To find a bifurcation boundary, Barnett and Duzhak (2014) first vary parameter α_2 , while holding all other parameters constant. They use the critical point of α_2 at 0.00125 to trace out the bifurcation boundary. After tracking the first period doubling bifurcation point, Barnett and Duzhak (2014) choose the second parameter, the risk aversion parameter, σ , to vary simultaneously with parameter α_2 . They find a period doubling bifurcation will occur for a very narrow set of parameters α_2 corresponding to a passive reaction to future inflation, in the close proximity of zero. Similarly, they find a period doubling point for parameter $\kappa = 3.725$. After choosing a second parameter, σ , to be varied, Barnett and Duzhak (2014) compute the period doubling bifurcation boundary. Parameter κ is a nonlinear function of the discount factor and the parameter responsible for the degree of price rigidity. It shows that the period doubling bifurcation will occur, when the economy is characterized by a high level of price stickiness. After analyzing further parameter combinations, Barnett and Duzhak (2014) find that a period doubling bifurcation is also possible for lower values of κ accompanied by very high values of the policy parameter, α_1 , which shows that an aggressive reaction of the central bank to future inflation will lead to a period doubling bifurcation.

6. Zellner's Marshallian Macroeconomic Model¹⁴

6.1. Introduction

¹⁴ This section is summarized from Banerjee, Barnett, Duzhak and Gopalan (2011).

This section describes Banerjee, Barnett, Duzhak, and Gopalan's (2011) bifurcation analysis of the Marshallian Macroeconomic Model. The Marshallian Macroeconomic Model (MMM) in Zellner and Israilevich (2005) is described by sectoral demand, supply, and entry/exit equations, as well as factor markets, the government, and a monetary sector added to complete the model. The explicitly formulated entry/exit behavior model in the MMM can be described by the equation $\frac{\dot{N}}{N} = \gamma'(\Pi - F^e)$; i.e. the growth rate of firms in the industry is proportional to the difference in current industry profitability, Π , and the long-run future profitability in the industry, F^e . The speed of adjustment is determined by the parameter γ' . With an entry/exit equation for each industry introduced in the model, Zellner and Israilevich (2005) describe the dynamics of the model in key variables, such as price and output at the sectoral as well as at the aggregate level. Varying some parameters would change the equilibria and could possibly cause changes in the nature of the equilibria, such as the number of solutions and the stability properties of the equilibria. Banerjee, Barnett, Duzhak, and Gopalan (2011) examine the model's characteristics, as well as the possibility of cyclical behavior through bifurcation analysis with respect to the entry/exit parameter F^e .

Banerjee, Barnett, Duzhak, and Gopalan (2011) show that a Hopf bifurcation exists within the theoretically feasible parameter space, giving rise to stable cycles, when taking F_1 from the entry-exit equation as the candidate for bifurcation parameter. Future work with that model could take several directions. One would be to introduce expectations into firms' future profitability. Another could be to introduce the money market and examine the possibility of other kinds of bifurcations with respect to government and monetary policy parameters.

6.2. The Model¹⁵

Banerjee, Barnett, Duzhak, and Gopalan (2011) consider a two sector, continuous time version of the Marshallian Macroeconomic Model (MMM) as outlined in Zellner and Israilevich (2005). Each sector is characterized by an aggregate output demand function, an aggregate supply function, and entry-exit modeling. Banerjee, Barnett, Duzhak, and Gopalan (2011) also include the government that collects taxes on output, purchases output from the two sectors

¹⁵ The model description is modified from Banerjee, Barnett, Duzhak, and Gopalan (2011).

and inputs from the factor markets. They exclude the presence of money markets from the model at this stage.

i. Output Demand

As noted in Banerjee, Barnett, Duzhak, and Gopalan (2011), the total demand for goods in the i th sector, $i = 1, 2$, is the sum of the demands from the government and the aggregate demand from households. Aggregate demand is thus given by

$$S_i = G_i + P_i^{1-\eta_{ii}} P_j^{\eta_{ij}} (S(1 - T^S))^{\eta_{is}}, \quad (6.1)$$

where G_i is the nominal government expenditure in sector i , $S = S_1 + S_2$ is the total income (nominal output), T^S is the tax rate, η_{ii} is the own price elasticity, η_{ij} is the cross price elasticity, and η_{is} is the income elasticity.

To express (6.1) in terms of growth rates, the aggregate demand for goods in each sector is the weighted sum of growth rates of demand from the government and households,

$$\hat{S}_i = g_i \hat{G}_i + (1 - g_i) [(1 - \eta_{ii}) \hat{P}_i + \eta_{ij} \hat{P}_j + \eta_{is} (\hat{S} + \hat{T}^{S'})], \quad (6.2)$$

where g_i is the ratio of government spending in sector i to total sales in sector i and $T^{S'} = 1 - T^S$. We use the hat over symbols to designate growth rate.

ii. Output Supply

There are N_i identical firms in the i th sector, each using a Cobb-Douglas type production function, $q_i = A_i^* L_i^\alpha K_i^\beta$, with $0 < \alpha_i, \beta_i < 1$, and $0 < \theta_i = 1 - \alpha_i - \beta_i < 1$, where q_i is the product of a neutral technological change, labor, and capital augmentation factors. The aggregate nominal profit-maximizing output supply of each sector i is given by $S_i =$

$N_i P_i^{\frac{1}{\theta_i}} \omega^{\frac{\alpha_i}{\theta_i}} r^{\frac{\beta_i}{\theta_i}}$, where P_i , ω , and r are the price, wage rate, and rental rate respectively.

Converting to growth rates, output supply becomes

$$\hat{S}_i = \hat{N}_i + \frac{1}{\theta_i} \hat{P}_i - \frac{\alpha_i}{\theta_i} \hat{\omega} - \frac{\beta_i}{\theta_i} \hat{r}. \quad (6.3)$$

iii. Entry/Exit

Banerjee, Barnett, Duzhak, and Gopalan (2011) consider the simplest form of the entry/exit equation proposed by Zellner and Israilevich (2005),

$$\hat{N}_i = \gamma_i[\Pi_i - F_i], \quad (6.4)$$

where $\Pi_i = \theta_i S_i$ is the current nominal aggregate industry profit for sector i , while $F_i > 0$ represents the aggregate long-run equilibrium profits in sector i , taking account of discounted entry costs. These parameters are considered by Banerjee, Barnett, Duzhak, and Gopalan (2011) to be time invariant. The coefficient, $\gamma_i > 0$, is the speed of adjustment for sector i . The larger the value of γ_i , the faster the adjustment is.

The interpretation of the entry/exit equation in Banerjee, Barnett, Duzhak, and Gopalan (2011) is that a positive departure from equilibrium profits F_i^e will attract new firms into the industry, while a negative departure will induce firms to leave the industry, given $\gamma_i > 0$.

iv. Government

According to Banerjee, Barnett, Duzhak, and Gopalan (2011), total nominal government expenditure, G , is the sum of expenditures in each of the two sectors, G_i , and its expenditure on labor, G_L , and capital, G_K . Zellner and Israilevich (2005) assume that G_i , for all $i = 1, 2, L, K$, grows at the same rate as G . Under this assumption, Banerjee, Barnett, Duzhak, and Gopalan (2011) propose that $G_i = \zeta_i G$, where ζ_i is the fraction of total government expenditure in the i th market. Thus in terms of growth rates, we have $\hat{G}_i = \hat{G}$.

The government collects a single uniform tax at the rate T^s on output. The tax revenue R is given by $R = T^s S$, which is expressed as $\hat{R} = \hat{T}^s + \hat{S}$ in terms of growth rate. The exogenously determined deficit/surplus, D , is defined as the government expenditures as a percentage of revenues, i.e. $D = \frac{G}{R}$. In terms of growth rate, we have

$$\hat{G} = \hat{D} + \hat{R} = \hat{D} + \hat{T}^s + \hat{S}. \quad (6.5)$$

v. Factor Markets

According to Banerjee, Barnett, Duzhak, and Gopalan (2011), the aggregate profit-maximizing factor demands from sector i are $L_i = \frac{\alpha_i S_i}{\omega}$ and $K_i = \frac{\beta_i S_i}{r}$. The government demand for labor and capital are $L_g = \frac{G_L}{\omega}$ and $K_g = \frac{G_K}{r}$ respectively. In terms of growth rates, the total demand for each factor is the weighted sum of growth rates of sectoral demands and the government demand for that factor, shown as below:

$$\frac{L_1}{L} \hat{L}_1 + \frac{L_2}{L} \hat{L}_2 + \frac{L_g}{L} \hat{L}_g = l_1 \hat{L}_1 + l_2 \hat{L}_2 + l_g \hat{L}_g, \quad (6.6)$$

$$\frac{K_1}{K} \hat{K}_1 + \frac{K_2}{K} \hat{K}_2 + \frac{K_g}{K} \hat{K}_g = k_1 \hat{K}_1 + k_2 \hat{K}_2 + k_g \hat{K}_g. \quad (6.7)$$

The dependence of the weights is given in Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011). According to Zellner and Israilevich (2005), $L = \left(\frac{\omega}{P}\right)^\delta \left(\frac{S}{P}\right)^{\delta_s}$ and $K = \left(\frac{r}{P}\right)^\phi \left(\frac{S}{P}\right)^{\phi_s}$, where δ (or ϕ) and δ_s (or ϕ_s) are price and income elasticities of labor (or capital). In terms of growth rates, the labor and capital supplies equal

$$\hat{L} = \delta(\hat{\omega} - \hat{P}) + \delta_s(\hat{S} - \hat{P}), \quad (6.8)$$

$$\hat{K} = \phi(\hat{r} - \hat{P}) + \phi_s(\hat{S} - \hat{P}). \quad (6.9)$$

vi. Quantity and Price Aggregates

The growth rates of aggregate nominal sales and the price aggregate are given by

$$\hat{S} = s_1 \hat{S}_1 + s_2 \hat{S}_2, \quad (6.10)$$

$$\hat{P} = s_1 \hat{P}_1 + s_2 \hat{P}_2, \quad (6.11)$$

where $s_i = \frac{S_i}{S}$.

vii. Solving the Model

The MMM model is solved using market clearing conditions in all markets and the government's flow budget identity. The complete solution procedure is outlined in Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011). All the equations in the model are reduced to yield the following two dynamic equations that govern the behavior of S_1 and S_2 :

$$\begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1(S_1, S_2; \boldsymbol{\Omega}) \\ \mathcal{F}_2(S_1, S_2; \boldsymbol{\Omega}) \end{bmatrix} = \mathcal{F}(S_1, S_2; \boldsymbol{\Omega}). \quad (6.12)$$

The explicit form of the non-linear functions, \mathcal{F}_1 and \mathcal{F}_2 , can be found in Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011). The vector $\boldsymbol{\Omega}$ consists of all structural parameters. The entry parameter for sector 1, F_1 , is taken as the bifurcation parameter in the following section. According to Appendix A in Banerjee, Barnett, Duzhak, and Gopalan (2011),

$$\mathcal{F}(S_1, S_2; \boldsymbol{\Omega}) = (\mathcal{H}(S_1, S_2; \boldsymbol{\Omega}))^{-1} \mathcal{D}(S_1, S_2; \boldsymbol{\Omega}), \quad (6.13)$$

where \mathcal{H} is a matrix of dimension 2×2 and \mathcal{D} is a vector of dimension 2×1 . The elements of \mathcal{H} and \mathcal{D} produce a high degree of nonlinearity in \mathcal{F} . In determining the dynamics of the equilibrium, several equilibria can arise.

To solve for an equilibrium, (S_1, S_2) , such that $\dot{S}_1 = 0$ and $\dot{S}_2 = 0$, it suffices to solve $\mathcal{F}(S_1, S_2; \boldsymbol{\Omega}) = 0$ in the system (6.12). From equation (6.13), the solutions at which $\mathcal{D} = 0$ will always be an equilibrium. Assuming there is no growth in government deficit, D , and taxes, T^s , the solution is based on (6.4), so that

$$S_1 = \frac{1}{\theta_1} F_1 \quad \text{and} \quad S_2 = \frac{1}{\theta_2} F_2. \quad (6.14)$$

The positive solutions are economically relevant and produce long run equilibrium by ensuring that there is no further entry/exit in either sector. The next section surveys Banerjee, Barnett, Duzhak, and Gopalan's (2011) results on stability and their bifurcation analysis of this equilibrium.

6.3. Stability and Bifurcation Analysis of Equilibrium

By generalizing the analysis of Veloce and Zellner's (1985) one sector MMM model to two sectors, Banerjee, Barnett, Duzhak, and Gopalan (2011) analyze the dynamics in terms of

convergence to the equilibrium given by (6.14). They consider the effects of cross price and income elasticities along with own price elasticities and emphasize two results that arise in the multisector model: (1) the solution may be stable, even when the two sectors have elastic demand; and (2) the path to the long run equilibrium may not be monotonic, so oscillatory damped convergence may arise.

Banerjee, Barnett, Duzhak, and Gopalan (2011) explain the occurrence of oscillatory convergence to equilibrium in terms of economic theory. They begin the analysis by assuming that the two sectors produce normal goods, which are substitutes and have elastic demand, and assuming Sector 1 is out of equilibrium, so that $S_1 > \frac{1}{\theta_1} F_1$, and $S_2 = \frac{1}{\theta_2} F_2$. Since $S_1 > \frac{1}{\theta_1} F_1$, current profitability is higher than equilibrium profitability, so entry takes place in Sector 1. The increase of supply in Sector 1 causes a drop in Sector 1's price, P_1 , and consequently causes sales, S_1 , having elastic demand, to increase. In addition, there is a decrease in Sector 2's demand, since the two goods are substitutes. There are two opposing effects on S_1 . If Sector 2's demand decreases, both Sector 2's price, P_2 , and quantity, Q_2 , decline, leading to a decline in Sector 2's sales, S_2 . If this decline in S_2 is greater in magnitude than the initial increase in S_1 , then $S = S_1 + S_2$ will decline, resulting in a fall in S_1 . Hence cross price and aggregate income effect may offset, having potentially destabilizing influence.

Banerjee, Barnett, Duzhak, and Gopalan (2011) further note that the decline in P_1 causes a decrease in Sector 2's demand and hence a decline in Sector 2's sales, which drop below the equilibrium, so that $S_2 < \frac{1}{\theta_2} F_2$. The result is an increase in S_2 and consequently an increase in S_1 through the income effect. Consequently the oscillatory convergence to equilibrium arises from interaction between the magnitudes of the shift and the elasticities. The mechanism depends largely on the own price, cross price, and income elasticities, and the magnitude of the shifts in demand and supply in each sector. Banerjee, Barnett, Duzhak, and Gopalan (2011) observe it is possible that the insufficiency of these shifts may result in the unstable solution, and they emphasize the importance of consistency between the elasticity parameters and the values of other parameters in production, input markets, entry/exit equations, and government

policy. The possibility exists that the economy could change its convergence type, if some of these parameters were to change.

Banerjee, Barnett, Duzhak, and Gopalan (2011) find the existence of a Hopf bifurcation, occurring when the Jacobian of \mathcal{F} has a pair of purely imaginary eigenvalues at some critical value of a bifurcation parameter. In the following analysis, they vary only parameter F_1 , while keeping all other parameters at values given in their paper's Appendix B. To analyze a codimension-1 Hopf bifurcation for the system (6.12), they first search for the value of (S_1, S_2) and the bifurcation parameter (F_1) satisfying the following conditions:

$$\mathcal{F}_1(S_1, S_2, F_1) = 0, \quad (6.15)$$

$$\mathcal{F}_2(S_1, S_2, F_1) = 0, \quad (6.16)$$

$$tr(\mathbf{J}_{\mathcal{F}}(S_1, S_2, F_1)) = 0, \quad (6.17)$$

$$det(\mathbf{J}_{\mathcal{F}}(S_1, S_2, F_1)) > 0, \quad (6.18)$$

where $\mathbf{J}_{\mathcal{F}}$ is the Jacobian of \mathcal{F} .

Banerjee, Barnett, Duzhak, and Gopalan (2011) observe that equations (6.15) and (6.16) yield the equilibrium for the system of differential equations in (6.12). Conditions (6.17) and (6.18) ensure that the eigenvalues of $\mathbf{J}_{\mathcal{F}}$ are purely imaginary. They find the existence of a Hopf bifurcation at the computed critical value $F^H = 6.070386762$ by verifying that conditions (6.17) and (6.18) are satisfied and the slope of the trace is not zero. Thus, as the parameter F_1 crosses F^H from the right, the solution given in (6.14) goes from a stable equilibrium to an unstable one. Banerjee, Barnett, Duzhak, and Gopalan (2011) illustrate that the system is locally spiraling inward for $F_1 > F^H$, and the system exhibits stable cycles in the phase space for F_1 close enough to F^H and $F_1 < F^H$.

7. Open-Economy New Keynesian Models¹⁶

¹⁶ This section is summarized from Barnett and Eryilmaz (2013,2014).

7.1. Introduction

The Barnett and Duzhak's (2008, 2010, 2013) results surveyed in sections 4 and 5 on bifurcation of New Keynesian models is based on closed economy models. Continuing to explore bifurcation in macroeconometric models, Barnett and Eryilmaz (2014) explore bifurcation of an open economy New Keynesian model proposed by Gali and Monacelli (2005). In addition, Barnett and Eryilmaz (2013) explore bifurcation of the open economy New Keynesian model proposed by Claridy, Gali, and Gertler (2002). In this section, we first survey the results of Barnett and Eryilmaz (2014) and then the results of Barnett and Eryilmaz (2013).

With those two models, Barnett and Eryilmaz (2013, 2014) find that the open economy framework has more complex dynamics than the closed economy models. As a result, stratification of the confidence regions remains an important research topic in the context of open-economy New Keynesian structures. In addition to damaging inference robustness, bifurcation of those models can result from changes in monetary policy. Such phenomena are relevant to evaluating policy risk.

As surveyed in section 7.2 below, Barnett and Eryilmaz (2014) ran bifurcation analyses of the Gali and Monacelli (2005) model and found that the degree of openness has a significant role in equilibrium determinacy and emergence of bifurcations. The values of bifurcation parameters and location of bifurcation boundaries are affected by introducing parameters related to the open economy structure. Numerical analyses are performed to search for different types of bifurcation. Limit cycles and period doubling bifurcations are found, although in some cases only for nonfeasible parameter values. Stratification of the confidence regions remains problematic to open economy New Keynesian functional structures.

Comparing the results from Barnett and Duzhak's (2010) closed economy analysis, it is not clear whether openness makes the New Keynesian model more sensitive to bifurcations. Barnett and Eryilmaz (2014) do not find evidence that open economies are more vulnerable to the problem than closed economies. The evidence from the Gali and Monacelli model might be caused by the model's broad set of parameters, including deep parameters relevant to the open economy. The fact that the studies use different sets of benchmark values for the parameters makes direct comparison more difficult. Barnett and Eryilmaz (2014) also note that the analysis is restricted to special cases within the framework of open-economy New

Keynesian structures. Generalizing the results to real economies would require more results with other open-economy New Keynesian models.

As surveyed in section 7.3 below, Barnett and Eryilmaz (2013) investigate bifurcations in the Clarida, Gali, and Gertler (2002) model. Barnett and Eryilmaz (2013) show that the model is vulnerable to Hopf bifurcation at a critical value of the parameter measuring the sensitivity of the nominal interest rate to changes in output gap. Their theoretical results need to be confirmed by subsequent numerical analysis to locate the Hopf bifurcation boundary and map its shape. The numerical analysis is beyond the scope of Barnett and Eryilmaz (2013), but they have provided the theory needed to implement the numerical research and locate the Hopf bifurcation boundary. A primary objective of the subsequent numerical analysis should be to determine whether the Hopf bifurcation boundary crosses relevant confidence regions of the model's parameters. If so, a serious robustness problem would exist in dynamical inferences. But even if the bifurcation boundary does not cross the confidence region, policy can move the location of the bifurcation boundary. Within this model, the central bank should react cautiously to changes in the rate of domestic inflation and the output gap to avoid inducing instability from a possible Hopf bifurcation.

7.2. Gali and Monacelli Model¹⁷

The Gali and Monacelli (2005) model is described by the following equations:

$$x_t = E_t x_{t+1} - \frac{1 + \alpha(\omega - 1)}{\sigma} (r_t - E_t \pi_{t+1} - \bar{r}_t), \quad (7.1)$$

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1 - \beta\theta)(1 - \theta)}{\theta} \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) x_t, \quad (7.2)$$

$$r_t = \bar{r}_t + \phi_\pi \pi_t + \phi_x x_t. \quad (7.3)$$

The Gali and Monacelli (2005) model is based on the following assumptions: the domestic policy does not affect the other countries or the world economy; each economy is assumed to have identical preferences, technology, and market structure; both consumers and

¹⁷ The model description is modified from Barnett and Eryilmaz (2014).

firms are assumed to behave optimally. Consumers maximize expected present value of utility, while firms maximize profits.

The utility maximization problem yields the dynamical intertemporal IS curve (7.1), which is a log-linear approximation to the Euler equation. In equation (7.1), x_t is the gap between actual output and flexible-price equilibrium output, \bar{r}_t is the small open economy's natural rate of interest, and $\sigma_\alpha = \sigma(1 - \alpha + \alpha\omega)^{-1}$ and $\omega = \sigma\gamma + (1 - \alpha)(\sigma\eta - 1)$ are composite parameters. The lowercase letters denote the logs of the respective variables, $\rho = \beta^{-1} - 1$ denote the time discount rate, and a_t is the log of labor's average product. The maximization problem of the representative firm yields the aggregate supply curve (7.2), also often called the New Keynesian Phillips curve in log-linearized form.

The policy rule (7.3) is a version of the Taylor rule, providing a simple (non-optimized) monetary policy, where the coefficients $\phi_x > 0$ and $\phi_\pi > 0$ measure the sensitivity of the nominal interest rate to changes in output gap and inflation rate respectively. Various versions of the Taylor rule are often employed to design monetary policy in empirical DSGE models. Equations (7.1) and (7.2), in combination with a monetary policy rule such as equation (7.3), constitute a small open economy model in the New Keynesian tradition.

Gali and Monacelli (2005) observed that closed economy models and open economy models differ in two primary aspects: (1) some coefficients, such as the degree of openness, terms of trade, and substitutability among domestic and foreign goods, depend on the parameters that are exclusive to the open economy framework; and (2), the natural levels of output and interest rate depend upon both domestic and foreign disturbances, in addition to openness and terms of trade. Barnett and Eryilmaz (2014) use the same methodology as in section 4 to detect bifurcation phenomenon. For two-dimensional dynamical systems, they apply Theorem 1.1. For three-dimensional dynamical systems, they apply Theorem 4.1. They employed CL MatCont within MatLab for numerical analysis. Regarding different policy rules, Barnett and Eryilmaz (2014) consider contemporaneous, forward, and backward looking policy rules, as well as hybrid combinations. The calibrated values of the parameters are given in Gali and Monacelli (2005), which are $\beta = 0.99$, $\alpha = 0.4$, $\sigma = \omega = 1$, $\varphi = 3$, and $\mu = 0.086$. For the $N = 3$ policy parameters, $\phi_x = 0.125$, $\phi_\pi = 1.5$, and $\phi_r = 0.5$.

i. Current-Looking Taylor Rule

The model consists of the following equations, in which the first two equations describe the economy, while the third equation is the monetary policy rule followed by the central bank with $N = 2$ policy parameters:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) x_t, \quad (7.4)$$

$$x_t = E_t x_{t+1} - \frac{1 + \alpha(\omega - 1)}{\sigma} (r_t - E_t \pi_{t+1} - \bar{r}_t), \quad (7.5)$$

$$r_t = \bar{r}_t + \phi_\pi \pi_t + \phi_x x_t. \quad (7.6)$$

Rearranging the terms, the system can be written in the form $E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t$,

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\mu}{\beta} + (1 + \alpha(\omega - 1)) \left(\frac{\beta \phi_x + \varphi \mu}{\beta \sigma} \right) & \frac{(\beta \phi_\pi - 1)(1 + \alpha(\omega - 1))}{\beta \sigma} \\ -\frac{\mu}{\beta} \left(\varphi + \frac{\sigma}{1 + \alpha(\omega - 1)} \right) & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}. \quad (7.7)$$

Using Theorem 1.1, the conditions for the existence of Hopf bifurcation in the system (7.7) are presented in the following proposition.

Proposition 7.1. Let Δ be the discriminant of the characteristic equation. Then system (7.7) undergoes a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\sigma(\beta - 1)}{1 + \alpha(\omega - 1)} - \mu \left(\varphi + \frac{\sigma}{1 + \alpha(\omega - 1)} \right) \phi_\pi. \quad (7.8)$$

The corresponding value of the bifurcation parameter in the closed economy case is $\phi_x^* = \sigma(\beta - 1) - \kappa \phi_\pi$, as given by Barnett and Duzhak (2008). For $\alpha = 0$, proposition 7.1 gives the same result as the closed economy counterpart.

Barnett and Eryilmaz (2014) numerically find a period doubling bifurcation at $\phi_x = -2.43$ and a Hopf bifurcation at $\phi_x = -0.52$. Numerical computations indicate that the

monetary policy rule equation (7.6) should have $\phi_x^* < 0$ for a Hopf or period doubling bifurcation to occur. That negative coefficient for the output gap in equation (7.6) would indicate a procyclical monetary policy: rising interest rates, when the output gap is negative, or vice versa. Literature seeking to explain procyclicality in monetary policy includes Schettkat and Sun (2009), Demirel (2010), and Leith, and Moldovan, and Rossi (2009). A successful countercyclical monetary policy would be bifurcation-free and would yield more robust dynamical inferences with confidence regions not crossing a bifurcation boundary.

Barnett and Eryilmaz (2014) also show there is only one periodic solution, while the other solutions diverge from the periodic solution as $t \rightarrow \infty$. This periodic solution is called an unstable limit cycle. The model is not subject to bifurcation within the feasible parameter space, when $\phi_x > 0$ and $\phi_\pi > 0$, although bifurcation is possible within the more general functional structure of system (7.7).

ii. Current-Looking Taylor Rule With Interest Rate Smoothing

The model consists of the equations (7.4) and (7.5), along with the following policy rule having $N = 3$ policy parameters:

$$r_t = \bar{r}_t + \phi_\pi \pi_t + \phi_x x_t + \phi_r r_{t-1}. \quad (7.9)$$

The system can be written in the form $E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t + \mathbf{d}_t$ as:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ E_t r_{t+1} \end{bmatrix} = \mathbf{C} \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix} + \begin{bmatrix} -\frac{1-\alpha+\alpha\omega}{\sigma} \bar{r}_t \\ 0 \\ E_t \bar{r}_{t+1} - \phi_x \bar{r}_t \frac{1-\alpha+\alpha\omega}{\sigma} \end{bmatrix} \quad (7.10)$$

with

$$\mathbf{y}_t = \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix},$$

$$\mathbf{C} =$$

$$\begin{bmatrix} \frac{\mu}{\beta} \left(1 + \varphi \frac{1 - \alpha + \alpha\omega}{\sigma}\right) + 1 & -\frac{1 - \alpha + \alpha\omega}{\sigma\beta} & \frac{1 - \alpha + \alpha\omega}{\sigma} \\ -\frac{\mu}{\beta} \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi\right) & \frac{1}{\beta} & 0 \\ \phi_x + \frac{\mu}{\beta} \left(1 + \varphi \frac{1 - \alpha + \alpha\omega}{\sigma}\right) \left(\phi_x \frac{1 - \alpha + \alpha\omega}{\sigma} - \phi_\pi\right) & -\frac{1}{\beta} \left(\phi_x \frac{1 - \alpha + \alpha\omega}{\sigma} - \phi_\pi\right) & \phi_r + \phi_x \frac{1 - \alpha + \alpha\omega}{\sigma} \end{bmatrix}$$

Assuming the system (7.10) has a pair of complex conjugate eigenvalues and a real-valued eigenvalue, the following proposition states the conditions for the system to undergo a Hopf bifurcation.

Proposition 7.2. The system (7.10) undergoes a Hopf bifurcation at equilibrium points, if and only if the following transversality condition holds

$$\frac{\partial |\lambda_i(\phi)|}{\partial \phi_j} \Big|_{\phi_j = \phi_j^*} \neq 0,$$

and also

$$(a) \quad \phi_r - \beta < 0, \tag{7.11}$$

$$(b) \quad \phi_r \left(\frac{\sigma(2 + \mu + 2\beta)}{1 - \alpha + \alpha\omega} + \varphi\mu \right) + \phi_x(\beta + 1) + \mu \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) (\phi_\pi + 1) + \frac{2\sigma}{1 + \alpha(\omega - 1)} < 0, \tag{7.12}$$

$$(c) \quad \phi_r^2 \xi_4 + \phi_r \xi_3 + (\phi_x \phi_r + \phi_x) \xi_2 + \phi_\pi \xi_1 + \xi_0 = -1. \tag{7.13}$$

Hopf bifurcation cannot occur in the model, since (7.12) does not hold. To detect the existence of a period doubling bifurcation, Barnett and Eryilmaz (2014) keep the structural parameters and policy parameters, ϕ_π and ϕ_r , constant at their baseline values, while varying the policy parameter ϕ_x over a feasible range. They numerically find period doubling bifurcation at $\phi_x = 0.83$. When they consider ϕ_π as the bifurcation parameter, they numerically find a period doubling bifurcation at $\phi_\pi = 5.57$ and a branching point at $\phi_\pi = -1.5$. There is no bifurcation of any type at $(\omega, \alpha) = (0, 1)$.

iii. Forward-Looking Taylor Rule

The model consists of equations (7.4) and (7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x E_t x_{t+1}. \quad (7.14)$$

Rearranging terms, the system can be written in the form

$$E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t, \quad (7.15)$$

with

$$\mathbf{y}_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\beta\sigma - (\mu\sigma + \mu\varphi(1 + \alpha(\omega - 1))) (\phi_\pi - 1)}{\beta\sigma - \beta\phi_x(1 + \alpha(\omega - 1))} & \frac{(\phi_\pi - 1)(1 + \alpha(\omega - 1))}{\beta\sigma - \beta\phi_x(1 + \alpha(\omega - 1))} \\ -\frac{\mu\sigma + \mu\varphi(1 + \alpha(\omega - 1))}{\beta + \alpha\beta(\omega - 1)} & \frac{1}{\beta} \end{bmatrix}.$$

Assuming the system (7.15) has a pair of complex conjugate eigenvalues, the following proposition provides the conditions for the system to undergo a Hopf bifurcation.

Proposition 7.3. The system (7.15) undergoes a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\beta - 1}{\beta} \frac{\sigma}{1 + \alpha(\omega - 1)} \quad (7.16)$$

Barnett and Eryilmaz (2014) find a period doubling bifurcation at $\phi_x = 1.913$ and a Hopf bifurcation at $\phi_x = -0.01$. Given the baseline values of the parameters, Hopf bifurcation occurs outside the feasible set of parameter values. There is no bifurcation at $(\alpha, \omega) = (1, 0)$. The system has a periodic solution at $\phi_\pi = 2.8$ and $\phi_x = 0$. The origin is a stable spiral point. Any solution that starts around the origin in the phase plane will spiral toward the origin. The origin is a stable sink, since the trajectories spiral inward.

iv. Pure Forward-Looking Inflation Targeting

The model consists of equations (7.4) and (7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi E_t \pi_{t+1}. \quad (7.17)$$

Rearranging the terms, the system can be written in the form

$$E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t, \quad (7.18)$$

with $\mathbf{y}_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}$,

$$\mathbf{C} = \begin{bmatrix} 1 - \left(\frac{\mu}{\beta} + \frac{\varphi \mu (1 + \alpha(\omega - 1))}{\beta \sigma} \right) (\phi_\pi - 1) & \frac{(\phi_\pi - 1)(1 + \alpha(\omega - 1))}{\beta \sigma} \\ -\frac{\mu}{\beta} \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) & \frac{1}{\beta} \end{bmatrix}.$$

Assuming the system (7.18) has a pair of complex conjugate eigenvalues, the following proposition provides the conditions for the system to undergo a Hopf bifurcation.

Proposition 7.4. The system (7.18) undergoes a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and $\beta^* = 1$. (7.19)

Barnett and Eryilmaz (2014) show that the solution path for $\beta = 1$ and $\phi_\pi = 8$ is periodic and oscillates around the origin, which is a stable center. Hopf bifurcation appears at $\beta = 1$ regardless of the values of α and ω . This result is the same as in the closed economy case under forward-looking inflation targeting in Barnett and Duzhak (2010). But setting the discount factor at 1 is not justifiable for a New Keynesian model, whether within an open or closed economy framework. Barnett and Eryilmaz (2014) also numerically find a period doubling bifurcation at $\beta = -0.91$, which is not theoretically feasible.

Barnett and Eryilmaz (2014) further show that there is only one periodic solution, which is an unstable limit cycle, and other solutions diverge from the periodic solution at $t \rightarrow \infty$. Varying ϕ_π while setting $\beta = 1$ and keeping the other parameters constant at their baseline values, they numerically find a Hopf bifurcation at $\phi_\pi = 1.0176$, a period doubling bifurcation at $\phi_\pi = 12.76$, and a branching point at $\phi_\pi = 1$.

v. Backward-Looking Taylor Rule

The model consists of equations (7.4) and (7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1}. \quad (7.20)$$

The system can be written in the form $E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t + \mathbf{d}_t$:

$$E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t + \begin{bmatrix} -\frac{1+\alpha(\omega-1)}{\sigma} \bar{r}_t \\ 0 \\ E_t \bar{r}_{t+1} \end{bmatrix}, \quad (7.21)$$

with

$$\mathbf{y}_t = \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\mu}{\beta} \left(1 + \frac{\varphi(1+\alpha(\omega-1))}{\sigma} \right) + 1 & -\frac{1+\alpha(\omega-1)}{\beta\sigma} & \frac{1+\alpha(\omega-1)}{\sigma} \\ -\frac{\mu}{\beta} \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) & \frac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & 0 \end{bmatrix}.$$

Assuming the system (7.21) has a pair of complex conjugate eigenvalues, the following proposition provides the conditions for the system to undergo a Hopf bifurcation.

Proposition 7.5. The system (7.21) undergoes a Hopf bifurcation at equilibrium points, if and only if the transversality condition, $\frac{\partial |\lambda_i(\Phi)|}{\partial \phi_j} \Big|_{\Phi=\Phi^*} \neq 0$, holds for some j ; and the following

conditions also are satisfied:

$$(i) \quad \phi_x + \phi_\pi \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) - \frac{\beta\sigma}{1+\alpha(\omega-1)} < 0, \quad (7.22)$$

$$(ii) \quad \phi_x(\beta - 1) + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) (1 - \phi_\pi) < 0, \quad (7.23)$$

$$(iii) \quad \left(\phi_x + \phi_\pi \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) \right)^2 + \left(\phi_x + \phi_\pi \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) \right) \xi_1$$

$$-\phi_x \xi_2 = \xi_3. \quad (7.24)$$

Barnett and Eryilmaz (2014) numerically find a period doubling bifurcation at $\phi_x = 1.91$. Starting from the point $\phi_x = 1.91$, they construct the period doubling boundary by varying ϕ_x and ϕ_π simultaneously. They also show that along the bifurcation boundary, the positive values of ϕ_x lie between 0 and 13. As the magnitude of ϕ_π increases, smaller values of ϕ_x would be sufficient to cause period doubling bifurcation under a backward-looking policy. Their numerical analysis with CL MatCont detects a codimension-2 fold-flip bifurcation (LPPD) at $(\phi_x, \phi_\pi) = (0.94, 2.01)$ and a flip-Hopf bifurcation (PDNS) at $(\phi_x, \phi_\pi) = (-6.98, 3.36)$. By treating the policy parameter ϕ_π as the potential source of bifurcation, numerical analysis also indicates a period doubling bifurcation at $\phi_\pi = 11.87$. By varying ϕ_π while keeping the other parameters constant at their benchmark values, another period doubling bifurcation is found at relatively large values of the parameter $\phi_\pi = 11.87$, which is nevertheless still within the feasible subset of the parameter space defined by Bullard and Mitra (2002).

vi. Backward-Looking Taylor Rule with Interest Rate Smoothing

The model consists of equations (7.4) and (7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1} + \phi_\pi r_{t-1}. \quad (7.25)$$

The system can be written in the form $E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t + \mathbf{d}_t$:

$$E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t + \begin{bmatrix} -\frac{1 + \alpha(\omega - 1)}{\sigma} \bar{r}_t \\ 0 \\ E_t \bar{r}_{t+1} \end{bmatrix}, \quad (7.26)$$

with

$$\mathbf{y}_t = \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\mu}{\beta} \left(1 + \frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right) + 1 & -\frac{1+\alpha(\omega-1)}{\beta\sigma} & \frac{1+\alpha(\omega-1)}{\sigma} \\ -\mu \left(1 + \frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right) & \frac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & \phi_r \end{bmatrix}.$$

Proposition 7.6. The system (7.26) undergoes a Hopf bifurcation at equilibrium points, if and only if the transversality condition, $\frac{\partial |\lambda_i(\Phi)|}{\partial \phi_j} \Big|_{\Phi=\Phi^*} \neq 0$, holds for some j ; and the following

conditions also are satisfied:

$$(i) \left| \frac{\phi_x - \phi_r \frac{\sigma}{1+\alpha(\omega-1)} + \phi_\pi \left(\frac{\sigma\mu}{1+\alpha(\omega-1)} + \varphi\mu \right)}{\frac{\beta\sigma}{1+\alpha(\omega-1)}} \right| < 1,$$

with $\phi_x - \phi_r \xi_2 + \phi_\pi \xi_3 < \frac{\beta\sigma}{1+\alpha(\omega-1)}$, and $\phi_r < \phi_x \xi_2 + \phi_\pi \xi_1 + \beta$,

$$(ii) \left| \phi_x \frac{1-\alpha+\alpha\omega}{\beta\sigma} - \phi_r \frac{1}{\beta} + \phi_\pi \mu \left(\frac{1}{\beta} + \varphi \frac{1-\alpha+\alpha\omega}{\beta\sigma} \right) - \left(\phi_r + \frac{1+\mu}{\beta} + \varphi\mu \frac{1-\alpha+\alpha\omega}{\beta\sigma} + 1 \right) \right| < 1 + \phi_r \left(\frac{1+\mu}{\beta} + \varphi\mu \frac{1-\alpha+\alpha\omega}{\beta\sigma} + 1 \right) - \phi_x \frac{1-\alpha+\alpha\omega}{\beta\sigma} + \frac{1}{\beta}$$

with $\phi_x \xi_2 + \phi_\pi \xi_1 - (1 + \phi_r) \xi_0 < 0$, and $\phi_x \xi_3 - \xi_4(\phi_\pi + \phi_r - 1) < 0$,

$$(iii) \phi_r \left(\frac{1+\mu}{\beta} + \varphi\mu \frac{1-\alpha+\alpha\omega}{\beta\sigma} + 1 \right) - \phi_x \frac{1-\alpha+\alpha\omega}{\sigma} + \frac{1}{\beta} + \left(\phi_x \frac{1-\alpha+\alpha\omega}{\beta\sigma} - \phi_r \frac{1}{\beta} + \phi_\pi \mu \left(\frac{1}{\beta} + \varphi \frac{1-\alpha+\alpha\omega}{\beta\sigma} \right) \right) \left(\phi_r + \frac{1+\mu}{\beta} + \varphi\mu \frac{1-\alpha+\alpha\omega}{\beta\sigma} + 1 \right) = 1 - \left(\phi_x \frac{1-\alpha+\alpha\omega}{\beta\sigma} - \phi_r \frac{1}{\beta} + \phi_\pi \mu \left(\frac{1}{\beta} + \varphi \frac{1-\alpha+\alpha\omega}{\beta\sigma} \right) \right)^2.$$

Barnett and Eryilmaz (2014) detect a period doubling bifurcation numerically at $\phi_x = 3$, given the benchmark values of the parameters and the setting $\phi_r = 0.5$. When $\phi_r = 1$, period doubling bifurcation occurs at $\phi_x = 4.09$. They find bifurcation boundary by varying ϕ_x and ϕ_π simultaneously, and then ϕ_x and ϕ_r simultaneously. In (ϕ_x, ϕ_π) -space, the bifurcation

boundary lies within the narrow range from $\phi_x = 3$ and $\phi_x = 3.25$. In contrast, ϕ_x varies more elastically in response to changes in ϕ_r along the bifurcation boundary in (ϕ_r, ϕ_x) -space.

Barnett and Eryilmaz (2014) further find codimension-2 fold-flip bifurcations at $(\phi_x, \phi_\pi) = (0.41, 3.19)$ and at $(\phi_x, \phi_r) = (0.78, -0.52)$, as well as flip-Hopf bifurcations at $(\phi_x, \phi_\pi) = (-10.44, 5.04)$ and $(\phi_x, \phi_r) = (-0.74, -1.23)$. Bifurcation disappears at $(\alpha, \omega) = (1, 0)$.

vii. Hybrid Taylor Rule

The model consists of equations (7.4) and (7.5) along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x x_t \quad (7.27)$$

The system can be written in the form:

$$E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t, \quad (7.28)$$

with

$$\mathbf{y}_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\beta \phi_x + \mu \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) (1 - \phi_\pi)}{\frac{\beta \sigma}{1 + \alpha(\omega - 1)}} + 1 & \frac{(\phi_\pi - 1)(1 + \alpha(\omega - 1))}{\beta \sigma} \\ -\frac{\mu}{\beta} \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) & \frac{1}{\beta} \end{bmatrix}.$$

Proposition 7.7. The system (7.28) exhibits a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\sigma(\beta - 1)}{1 + \alpha(\omega - 1)}. \quad (7.29)$$

Barnett and Eryilmaz (2014) find a period doubling bifurcation at $\phi_x = -1.92$ as well as a Hopf bifurcation at $\phi_x = -0.01$, while system parameters are at benchmark values.

Assuming positive values for policy parameters, values of the bifurcation parameters are outside the feasible region of the parameter space. They conclude that the feasible set of parameter values for ϕ_x does not include a bifurcation boundary. They also find that in the (ϕ_π, ϕ_x) -space, along the period-doubling bifurcation boundary, the bifurcation parameter ϕ_x varies in the same direction as ϕ_π . Therefore as ϕ_x increases, higher values of ϕ_x are required to cause a period doubling bifurcation. They analyze the solution paths from (7.28) with stability properties indicating Hopf bifurcation. The inner spiral trajectory is converging to the equilibrium point, while the outer spiral is diverging.

7.3. Clarida, Gali, and Gertler Model

Barnett and Eryilmaz (2013) conduct bifurcation analysis in the open-economy New Keynesian model developed by Clarida, Gali, and Gertler (2002). Clarida, Gali, and Gertler (2002) developed a two-country version of a small open economy model, based on Clarida, Gali, and Gertler (2001) and Gali and Monacelli (1999).

Following Walsh (2003, pp.539-540), the model of Clarida, Gali, and Gertler (2002) can be written as follows:

$$\pi_t^h = \beta E_t \pi_{t+1}^h + \delta \left[\sigma + \eta + \left(\frac{v\sigma}{1+w} \right) \right] x_t, \quad (7.30)$$

$$x_t = E_t x_{t+1} - \left(\frac{1+w}{\sigma} \right) (r_t - E_t \pi_{t+1}^h - \bar{r}_t), \quad (7.31)$$

$$r_t = \bar{r}_t + \phi_\pi \pi_t^h + \phi_x x_t. \quad (7.32)$$

Equation (7.30) is an inflation adjustment equation for the aggregate price of domestically produced goods. Equation (7.31) is the dynamic IS curve, derived from the Euler condition of the consumers' optimization problem. The monetary policy rule, (7.32), is a domestic-inflation-based current-looking Taylor rule.

Let x_t denote the output gap, π_t^h the inflation rate for domestically produced goods and services, and r_t the nominal interest rate, with E_t being the expectation operator and \bar{r}_t denoting the small open economy's natural rate of interest. The lowercase letter denotes the

logs of the respective variables. The coefficients $\phi_x > 0$ and $\phi_\pi > 0$ are the policy parameters, measuring the sensitivity of the nominal interest rate to changes in output gap and inflation rate, respectively. In addition, $\delta = [(1 - \theta)(1 - \beta\theta)]/\theta$ is a composite parameter with θ representing the probability that a firm holds its price unchanged in a given period of time, while $1 - \theta$ is the probability that a firm resets its price. The parameter η denotes the wage elasticity of labor demand, and σ^{-1} denotes the elasticity of intertemporal substitution. The parameter w denotes the growth rate of nominal wages, $\rho = \beta^{-1} - 1$ is the time discount rate, and v is the population size in the foreign country, with $1 - v$ being the population size of the home country. Wealth effect is captured by the term $v\sigma$.¹⁸

Substituting (7.32) for $r_t - \bar{r}_t$ into the equation (7.31), Barnett and Eryilmaz (2013) reduce the system to a first order dynamical system in two equations for domestic inflation and output gap. The system is given by:

$$\begin{aligned}\pi_t^h &= \beta E_t \pi_{t+1}^h + \delta \left[\sigma + \eta + \left(\frac{v\sigma}{1+w} \right) \right] x_t, \\ x_t &= E_t x_{t+1} - \left(\frac{1+w}{\sigma} \right) (\phi_\pi \pi_t^h + \phi_x x_t - E_t \pi_{t+1}^h).\end{aligned}$$

An equilibrium solution to the system is $x_t = \pi_t^h = 0$ for all t . The system can be written in the standard form as

$$\mathbf{A}E_t \mathbf{y}_{t+1} = \mathbf{B} \mathbf{y}_t, \quad (7.33)$$

or $E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t$, where $\mathbf{C} = \mathbf{A}^{-1} \mathbf{B}$, as follows:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1}^h \end{bmatrix} = \mathbf{C} \begin{bmatrix} x_t \\ \pi_t^h \end{bmatrix}, \quad (7.34)$$

where

$$\mathbf{C} = \begin{bmatrix} 1 + \frac{(1+w)\phi_x}{\sigma} + \delta(1+w) \left(\sigma + \eta + \left(\frac{v\sigma}{1+w} \right) \right) \frac{1}{\beta\sigma} & \frac{(1+w)\phi_\pi}{\sigma} - \frac{(1+w)}{\beta\sigma} \\ -\delta \left(\sigma + \eta + \left(\frac{v\sigma}{1+w} \right) \right) \frac{1}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

¹⁸ The model description is modified from Barnett and Eryilmaz (2013).

Assuming a pair of complex conjugate eigenvalues, the conditions for the existence of a Hopf bifurcation are provided in the following proposition.

Proposition 7.8. Let Δ be the discriminant of the characteristic equations. Then the system (7.34) undergoes a Hopf bifurcation at equilibrium points, if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right). \quad (7.35)$$

Proof. See Barnett and Eryilmaz (2013), Proposition 1.

Barnett and Eryilmaz (2013) observe that the Clarida, Gali, and Gertler (2002) model differs in several aspects from the Gali and Monacelli (2005) model. The degree to which the two models differ depends upon the parameter settings. In the Clarida, Gali, and Gertler (2002) model, the parameters w , v , and δ play an important role in determining the critical value of the bifurcation parameter. Barnett and Eryilmaz (2013) note that numerical implementation of the theory to locating Hopf bifurcation boundaries in the Clarida, Gali, and Gerler (2002) model would be a challenging project.

8. Two Endogenous Growth Models¹⁹

8.1. Introduction

This section surveys Barnett and Ghosh (2013, 2014) about bifurcation analyses of two endogenous growth models. Previous stability analyses of endogenous growth models include the following. Benhabib and Perli (1994) analyzed the stability property of the long-run equilibrium in the Lucas (1988) model; Arnold (2000a, 2000b) analyzed the stability of equilibrium in the Romer (1990) model; Arnold (2006) has done the same for the Jones (1995) model; and Mondal (2008) examined the dynamics of the Grossman-Helpman (1991) model of endogenous product cycles. The results derived in those papers provide important insights to researchers. But a detailed bifurcation analysis had not been provided for many of these popular endogenous growth models. Barnett and Ghosh (2014) filled the gap for the Uzawa-Lucas endogenous growth model, as surveyed in section 8.2 below, while Barnett and Ghosh

¹⁹ This section is summarized from Barnett and Ghosh (2013, 2014).

(2013) do so for a variant of Jones (2002) semi-endogenous growth model, as surveyed in section 8.3 below.

In section 8.2, Barnett and Ghosh (2014) conduct bifurcation analysis on the Uzawa-Lucas endogenous growth model, which is solved from a centralized social planner perspective as well as in the model's decentralized market economy form. Barnett and Ghosh (2014) locate transcritical bifurcation and Hopf bifurcation boundaries for the decentralized version of the model using Mathematica, and also investigate the existence of Hopf bifurcation, branch point bifurcation, limit point cycle bifurcation, and period doubling bifurcations using Matcont. The series of period doubling bifurcations confirm the existence of global bifurcation and reveal the possibility of chaotic dynamics. Barnett and Ghosh (2014) also point out that the externality of the human capital parameter plays an important role in determining the dynamics of the decentralized model. On the contrary, from the centralized social planner perspective, the solution is saddle path stable with no possibility of bifurcation within the feasible parameter set.

In section 8.3, Barnett and Ghosh (2013) conduct bifurcation analysis on a variant of the Jones (2002) model. Jones found that long-run growth arises from the worldwide discovery of ideas, which depend on the rate of population growth of the countries contributing to world research rather than on the level of population. His model exhibits "weak" scale effect, in contrast with the "strong" scale effect, produced by the first generation endogenous growth models of Romer (1990) and Grossman and Helpman (1991). Barnett and Ghosh (2013) incorporate human capital accumulation into a Jones model. They also consider the possibility that the direction of technology progress is driven by human capital investment (Bucci (2008)). As a result, the parameters in the human capital accumulation equation play an important role in determining the dynamics of the model. Barnett and Ghosh (2013) also introduce the possibility of decreasing returns to scale associated with human capital and with time spent accumulating human capital in the production equation. This assumption accounts for the scale effects in the model and permits a closed form solutions for the steady state of the model. Using the numerical package Matcont, Barnett and Ghosh (2013) further show the existence of Hopf bifurcation, branch point bifurcation, limit point of cycles, Bogdanov-Takens bifurcation, and generalized Hopf bifurcations within the feasible parameter sets.

In both models, Barnett and Ghosh (2013, 2014) emphasize that bifurcation boundaries do not necessarily separate stable from unstable solution domains. Barnett and Ghosh (2013, 2014) note that bifurcation boundaries can separate one kind of unstable dynamics domain from another kind of unstable dynamics domain. Not as well known is that bifurcation boundaries can separate one kind of stable dynamics domain from another kind of stable dynamics domain (called soft bifurcation), such as bifurcation from monotonic stability to damped periodic stability or from damped periodic to damped multiperiodic stability. Recognizing there are an infinite number of kinds of unstable dynamics as well as an infinite number of kinds of stable dynamics, subjective prior views on the stability of economies are not reliable without conducting analysis of model dynamics.

8.2. Uzawa-Lucas Endogenous Growth Model²⁰

The Uzawa-Lucas endogenous growth model (Uzawa (1965) and Lucas (1988)) is one of the most important endogenous growth models. This model has two sectors: the human capital production sector and the physical capital production sector, producing human capital and physical capital, respectively. Individuals have the same level of work qualification and expertise (H). They allocate some of their time to producing final goods and dedicate the remaining time to training and studying. Barnett and Ghosh (2014) solve the model from a centralized social planner perspective as well as from the model's decentralized market economy form.

The production function in the physical sector is defined as follows:

$$Y = AK^\alpha(\varepsilon hL)^{1-\alpha}h_a^\zeta, \quad 0 < \alpha < 1,$$

where Y is output, A is technology level, K is physical capital, α is the share of physical capital, L is labor, and h is human capital per person. In addition, ε and $1 - \varepsilon$ are respectively the fraction of labor time devoted to producing output and human capital, where $0 < \varepsilon < 1$. Observe that εhL is the quantity of labor, measured in efficiency units, employed to produce output, and h_a^ζ measures the externality associated with average human capital of the work

²⁰ The model description is modified from Barnett and Ghosh (2014).

force h_a , where ζ is the positive externality parameter in the production of human capital. In per capita terms, $y = Ak^\alpha(\varepsilon h)^{1-\alpha}h_a^\zeta$.

The physical capital accumulation equation is

$$\dot{K} = AK^\alpha(\varepsilon hL)^{1-\alpha}h_a^\zeta - C - \delta K.$$

In per capita terms, the equation is

$$\dot{k} = Ak^\alpha(\varepsilon h)^{1-\alpha}h_a^\zeta - c - (n + \delta)k,$$

and the human capital accumulation equation is

$$\dot{h} = \eta h(1 - \varepsilon),$$

where η is defined as schooling productivity.

The decision problem is

$$\max_{c_t, \varepsilon_t} \int_t^\infty \frac{e^{-(\rho-n)t} (c(\tau)^{1-\sigma} - 1)}{1-\sigma} dt \quad (8.1)$$

subject to

$$\dot{k} = Ak^\alpha(\varepsilon h)^{1-\alpha}h_a^\zeta - c - (n + \delta)k \quad (8.2)$$

and

$$\dot{h} = \eta(1 - \varepsilon)h, \quad (8.3)$$

where ρ ($\rho > n > 0$) is the subjective discount rate, and $\sigma \geq 0$ is the inverse of the intertemporal elasticity of substitution in consumption.

i. Social Planner Problem

The social planner takes into account the externality associated with human capital, when solving the maximization problem (8.1) subject to (8.2) and (8.3). From the first order conditions, Barnett and Ghosh (2014, Appendix 2) derive the equations describing the economy of the Uzawa-Lucas model from a social planner's perspective:

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n + \delta),$$

$$\frac{\dot{h}}{h} = \eta(1 - \varepsilon),$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho + \delta)}{\sigma},$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta \frac{(1 - \alpha + \zeta)}{1 - \alpha} \varepsilon + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - \frac{c}{k} + \frac{(1 - \alpha)}{\alpha} (n + \delta),$$

$$\frac{\dot{L}}{L} = n.$$

Let $m = \frac{Y}{K}$ and $g = \frac{c}{k}$. Taking logarithms of m and g and differentiating with respect to

time, the dynamics of the Uzawa-Lucas model is given by equation (8.4) and (8.5):

$$\frac{\dot{m}}{m} = -(1 - \alpha)m + \frac{1 - \alpha}{\alpha} (n + \delta) + \eta \frac{(1 - \alpha + \zeta)}{\alpha}. \quad (8.4)$$

$$\frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma} - 1\right) m - \frac{\rho}{\sigma} - \delta \left(\frac{1}{\sigma} - 1\right) + g + n. \quad (8.5)$$

The steady state (m^*, g^*) is given by $\dot{m} = \dot{g} = 0$ and is derived to be

$$m^* = \eta \frac{(1 - \alpha + \zeta)}{\alpha} + \frac{(n + \delta)}{\alpha},$$

$$g^* = \frac{\rho - n}{\sigma} + \frac{1 - \alpha}{\alpha} (n + \delta) + \eta \frac{(1 - \alpha + \zeta)(\sigma - \alpha)}{\alpha(1 - \alpha)\sigma}.$$

A unique steady state exists, if

$$\Lambda = \frac{(1 - \alpha + \zeta)}{\alpha} (\sigma - 1)\eta(1 - \varepsilon) + \rho > 0.$$

This inequality condition for Λ is the transversality condition for the consumer's utility maximization problem, as shown in Barnett and Ghosh (2014, Appendix 1). It can be shown that the social planner solution is saddle path stable. See, e.g., Barro and Sala-i-Martin (2003) and

Mattana (2004). Linearizing around the steady state, $s^* = (m^*, g^*)$, the local stability properties of the system defined by equations (8.4) and (8.5) can be found. The result is

$$\begin{bmatrix} \dot{m} \\ \dot{g} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \dot{m}}{\partial m} |_{s^*} & \frac{\partial \dot{m}}{\partial g} |_{s^*} \\ \frac{\partial \dot{g}}{\partial m} |_{s^*} & \frac{\partial \dot{g}}{\partial g} |_{s^*} \end{bmatrix}}_{\mathbf{J}_s} \begin{bmatrix} m_t - m^* \\ g_t - g^* \end{bmatrix},$$

where

$$\mathbf{J}_s = \begin{bmatrix} -(1 - \alpha)m^* & 0 \\ \left(\frac{\alpha}{\sigma} - 1\right)g^* & g^* \end{bmatrix}.$$

Since $m^* > 0$ and $g^* > 0$, it follows that $\det(\mathbf{J}_s) = -(1 - \alpha)m^*g^* < 0$. Hence the saddle path is stable.

ii. Representative Agent Problem

From the first order conditions with $h = h_a$, Barnett and Ghosh (2014, Appendix 3) derive the following equations describing the dynamics of the decentralized Uzawa-Lucas model:

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n + \delta),$$

$$\frac{\dot{h}}{h} = \eta(1 - \varepsilon),$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho + \delta)}{\sigma},$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta \frac{(\alpha - \zeta)}{1 - \alpha} \varepsilon + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - \frac{c}{k} + \frac{(1 - \alpha)}{\alpha} (n + \delta),$$

$$\frac{\dot{L}}{L} = n.$$

Taking logarithms of m and g and differentiating with respect to time, the following three equations define the dynamics of the Uzawa-Lucas model

$$\frac{\dot{m}}{m} = -(1-\alpha)m + \frac{(1-\alpha)}{\alpha}(n+\delta) + \eta \frac{(1-\alpha+\zeta)}{\alpha} - \eta \frac{\zeta}{\alpha} \varepsilon, \quad (8.6)$$

$$\frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma} - 1\right) m - \frac{\rho}{\sigma} - \delta \left(\frac{1}{\sigma} - 1\right) + g + n, \quad (8.7)$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta \frac{(\alpha-\zeta)}{\alpha} \varepsilon + \eta \frac{(1-\alpha+\zeta)}{\alpha} - g + \frac{(1-\alpha)}{\alpha} (n+\delta). \quad (8.8)$$

The steady state $(m^*, g^*, \varepsilon^*)$, given by $\dot{m} = \dot{g} = \dot{\varepsilon} = 0$, is

$$\varepsilon^* = 1 - \frac{(1-\alpha)(\rho - n - \eta)}{\eta[\zeta - \sigma(1-\alpha + \zeta)]}$$

$$m^* = \eta \frac{[1-\alpha + \zeta(1-\varepsilon^*)]}{\alpha(1-\alpha)} + \frac{n}{\alpha}$$

$$g^* = \eta \frac{[1-\alpha + \zeta(1-\varepsilon^*) + \alpha\varepsilon^*]}{\alpha(1-\alpha)} + \frac{n(1-\alpha)}{\alpha}$$

A unique steady state exists, if

$$\Lambda = \frac{(1-\alpha+\zeta)}{\alpha} (\sigma-1)\eta(1-\varepsilon) + \rho > 0,$$

and $0 < \varepsilon < 1$.

The inequality condition on Λ is the transversality condition for the consumer's utility maximization problem (Barnett and Ghosh (2014), appendix 1), while $0 < \varepsilon^* < 1$ is necessary for $m^*, g^* > 0$. Linearizing the system around the steady state, $s^* = (m^*, g^*, \varepsilon^*)$, yields the following:

$$\begin{bmatrix} \dot{m} \\ \dot{g} \\ \dot{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \dot{m}}{\partial m} |_{s^*} & \frac{\partial \dot{m}}{\partial g} |_{s^*} & \frac{\partial \dot{m}}{\partial \varepsilon} |_{s^*} \\ \frac{\partial \dot{g}}{\partial m} |_{s^*} & \frac{\partial \dot{g}}{\partial g} |_{s^*} & \frac{\partial \dot{g}}{\partial \varepsilon} |_{s^*} \\ \frac{\partial \dot{\varepsilon}}{\partial m} |_{s^*} & \frac{\partial \dot{\varepsilon}}{\partial g} |_{s^*} & \frac{\partial \dot{\varepsilon}}{\partial \varepsilon} |_{s^*} \end{bmatrix}}_{\mathbf{J}_m} \begin{bmatrix} m_t - m^* \\ g_t - g^* \\ \varepsilon_t - \varepsilon^* \end{bmatrix},$$

where

$$\mathbf{J}_m = \begin{bmatrix} -(1-\alpha)m^* & 0 & -\eta \frac{\zeta}{\alpha} m^* \\ \left(\frac{\alpha}{\sigma} - 1\right)g^* & g^* & 0 \\ 0 & -\varepsilon^* & \eta \frac{(\alpha - \zeta)}{\alpha} \varepsilon^* \end{bmatrix}.$$

The characteristic equation associated with \mathbf{J}_m is $q^3 + c_2q^2 + c_1q + c_0 = 0$, where

$$c_0 = \eta \frac{[\sigma(1-\alpha+\zeta) - \zeta]}{\sigma} m^* g^* \varepsilon^*,$$

$$c_1 = \eta^2 \frac{(\alpha - \zeta)}{\alpha} \varepsilon^{*2} - (1-\alpha)m^* g^*,$$

$$c_2 = -\eta \frac{(2\alpha - \zeta)}{\alpha} \varepsilon^*.$$

iii. Bifurcation Analysis

Barnett and Ghosh (2014) analyze the existence of codimension 1 and 2, transcritical, and Hopf bifurcation in the system ((8.6), (8.7),(8.8)). They search for the bifurcation boundary according to $c_0 = \det(\mathbf{J}_m) = 0$.

Theorem 8.1. \mathbf{J}_m has zero eigenvalues, if

$$\eta \frac{[\sigma(1-\alpha+\zeta) - \zeta]}{\sigma} m^* g^* \varepsilon^* = 0. \quad (8.9)$$

It follows from the Hopf Bifurcation Theorem in Guckenheimer and Holmes (1983), that if $c_0 - c_1c_2 = 0$ and $c_1 > 0$, then \mathbf{J}_m has precisely one pair of purely imaginary eigenvalues.

But if $c_0 - c_1 c_2 \neq 0$ and $c_1 > 0$, then \mathbf{J}_m has no purely imaginary eigenvalues. Therefore, Barnett and Ghosh (2014) derive the following theorem:

Theorem 8.2. The matrix \mathbf{J}_m has precisely one pair of pure imaginary eigenvalues, if

$$\left\{ \begin{array}{l} \alpha m^* g^* ((\alpha - 1)\alpha\sigma + \zeta(\sigma - \alpha)) + \eta^2 \sigma \varepsilon^{*2} (2\alpha - \zeta)(\alpha - \zeta) = 0, \\ \text{and} \\ \frac{\eta^2}{\alpha} \varepsilon^{*2} (\alpha - \zeta) - (1 - \alpha)m^* g^* > 0. \end{array} \right. \quad (8.10)$$

Furthermore, Barnett and Ghosh (2014) explain cyclical behavior in the model. They state that the increase of ζ would bring about the increase of savings rate since consumers are willing to cut current consumption in exchange for higher future consumptions. Then the movement of labor from output production to human capital production brings an increase in human capital, and subsequently faster accumulation of physical capital, if sufficient externality to human capital in production of physical capital is present. On the other hand, a lower subjective discount rate, ρ , could cause consumption to rise gradually with faster capital accumulation. This leads to greater consumption-goods production in the future, which eventually leads to a decline in savings rate. A cyclical convergence to equilibrium comes from these two opposing effects, when savings rate is different from the equilibrium rate. Barnett and Ghosh (2014) conclude that interaction between different parameters can cause cyclical convergence to equilibrium or may cause instability, and for some parameter values convergence to cycles may occur.

Based on Benhabib and Perli (1994), Barnett and Ghosh (2014) locate bifurcation boundaries by keeping some parameters free, while setting the others fixed at $\mathfrak{D}^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.65, 0.0505, 0.15, 0, 0)$ or $\mathfrak{W}^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.75, 0.0505, 0.15, 0, 0)$. Using Matcont, Barnett and Ghosh (2014) then investigate the stability properties of cycles generated by different combinations of parameters. Some limit cycles, such as supercritical bifurcations, are stable, while some other limit cycles, such as subcritical bifurcations, are unstable. A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Period doubling bifurcation occurs, when a new limit cycle, the period of which is twice that of the old one, emerges from an existing limit cycle.

Table 8.1 reports the values of the share of capital, α , the externality in production of human capital, ζ , and the inverse of the intertemporal elasticity of substitution in consumption, σ ²¹. Since each of the cases reported in Table 8.1 has positive first Lyapunov coefficient, an unstable limit cycle (i.e., periodic orbit) bifurcates from the equilibrium.

When α is the free parameter, Barnett and Ghosh (2014) find from continuing computation of limit cycles from the Hopf point, that two limit cycles with different periods are present near the limit point cycle (LPC) point at $\alpha = 0.738$. Continuing computation further, a series of period doubling (flip) bifurcations arise. The first period doubling bifurcation at $\alpha = 0.7132369$ has positive normal form coefficients, while the other period doubling bifurcations have negative normal form coefficients. This indicates that the first period doubling bifurcation has unstable double-period cycles, while the rest have stable double-period cycles. Barnett and Ghosh (2014) also find that the limit cycle approaches a global homoclinic orbit, which is a dynamical system trajectory joining a saddle equilibrium point to itself. They also point out the possibility of reaching chaotic dynamics through a series of period doubling bifurcation.

When ζ and σ are free parameters, Barnett and Ghosh (2014) conduct the bifurcation analysis in a similar way by carrying out the continuation of the limit cycle from the first Hopf point. They find that both cases give rise to the LPC point with a nonzero normal form coefficient, indicating the existence of a fold bifurcation at the LPC point.

²¹ Table 8.1 is a replicate of Barnett and Ghosh (2014) Table 1.

Table 8.1. Stability Analysis Of Uzawa-Lucas Growth Model

Parameters	Equilibrium Bifurcation	Bifurcation of Limit Cycle
<p>α</p> <p>Other parameters set at ϑ^*</p>	<p>Hopf (H)</p> <p>First Lyapunov coefficient = 0.00242, $\alpha=0.738207$</p>	<p>Limit Point Cycle (LPC)</p> <p>period= 231.206, $\alpha=0.7382042$,normal form coefficient=0.007</p> <p>Period Doubling (PD)</p> <p>period= 584.064, $\alpha=0.7132369$,normal form coefficient=0.910</p> <p>Period Doubling (PD)</p> <p>period= 664.005, $\alpha=0.7132002$,normal form coefficient=-0.576</p> <p>Period Doubling (PD)</p> <p>period= 693.988, $\alpha=0.7131958$,normal form coefficient=-0.469</p> <p>Period Doubling (PD)</p> <p>period= 713.978, $\alpha=0.7131940$,normal form coefficient=-0.368</p> <p>Period Doubling (PD)</p> <p>period= 725.667, $\alpha=0.7131932$,normal form coefficient=-0.314</p> <p>Period Doubling (PD)</p> <p>period= 784.104, $\alpha=0.7131912$,normal form coefficient=-0.119</p>
<p>ζ</p> <p>Other parameters set at ω^*</p>	<p>Hopf (H)</p> <p>First Lyapunov coefficient =0.00250,$\zeta=0.107315$</p> <p>Hopf (H)</p> <p>First Lyapunov coefficient =0.00246,$\zeta=0.047059$</p> <p>Branch Point (BP)</p> <p>$\zeta=0.047059$</p>	<p>Limit Point Cycle (LPC)</p> <p>period= 215.751, $\zeta=0.1073147$,normal form coefficient=0.009</p>
<p>σ</p> <p>Other parameters set at ω^*</p>	<p>Hopf (H)</p> <p>First Lyapunov coefficient =0.00264, $\sigma=0.278571$</p> <p>Hopf (H)</p> <p>First Lyapunov coefficient =0.00249, $\sigma=0.13939$</p> <p>Branch Point (BP)</p> <p>$\sigma=0.278571$</p>	<p>Limit Point Cycle (LPC)</p> <p>Period= 213.83, $\sigma=0.1394026$,normal form coefficient=0.009</p>

8.3. Jones Semi-Endogenous Growth Model²²

The model is based on a variant of Jones' (2002) semi-endogenous growth model.

The labor endowment equation is given by

$$L_{A_t} + L_{Y_t} = L_t = \varepsilon_t N_t, \quad (8.11)$$

where at time t , L_t is employment, L_{Y_t} is the labor employed in producing output, L_{A_t} is the total number of researchers, and N_t is the total population having rate of growth $n > 0$. Each person is endowed with one unit of time and divides the time among producing goods, producing ideas and human capital, while ε_t and $1 - \varepsilon_t$ represent respectively the amount of time the person spends producing output and accumulating human capital.

The capital accumulation equation is given by

$$\dot{K} = s_{k_t} Y_t - dK_t, \quad K_0 > 0, \quad (8.12)$$

and
$$\dot{K} = Y_t - C_t - dK_t, \quad (8.13)$$

where s_{k_t} is the fraction of output invested, d is the exogenous, constant rate of depreciation, Y_t is the aggregate production of homogenous final goods, and K_t is capital stock.

Output is produced using the total quantity of human capital, H_{Y_t} , and a set of intermediate goods. The total quantity of human capital equation is given by

$$H_{Y_t} = h_t L_{Y_t}, \quad (8.14)$$

with the individual's human capital accumulation equation is given by

$$\dot{h}_t = \eta h_t^{\beta_1} (1 - \varepsilon_t)^{\beta_2} - \theta g_A h_t, \quad 0 < \beta_1, \beta_2, \varepsilon_t < 1, \eta > 0, 1 + \theta > 0, \quad (8.15)$$

where h_t is human capital per person and L_{Y_t} is labor employed in producing output. The parameter η is productivity of human capital in the production of new human capital, θ reflects the effect of technological progress on human capital investment, and $g_A = \frac{\dot{A}}{A}$ is the growth

²² The model description is modified from Barnett and Ghosh (2013)

rate of technology. Equation (8.15) builds on the human capital accumulation equation from the Uzawa-Lucas model.

As noted in Barnett and Ghosh (2013), the human capital accumulation equation has two advantages. It accounts for the scale effects present in the model, and it makes the model tractable to solve for possible steady states. To see this, Barnett and Ghosh (2013) introduced the assumption of decreasing returns to scale of the human capital growth rate in (8.15) by setting $0 < \beta_1$ and $\beta_2 < 1$. The higher the level of human capital or of time spent accumulating human capital, the more difficult it is to generate additional human capital. If β_1 or β_2 is equal to 1, the model will exhibit “strong” scale effects. In models associated with strong scale effects, the growth rate of the economy is an increasing function of the population. But this phenomenon is inconsistent with United States data, as shown by Jones (1995). Barnett and Ghosh (2013) also include the technological growth rate, g_A , which directly influences the human capital growth rate. As in Bucci (2008), Barnett and Ghosh (2013) restrict $\theta > -1$ to prevent explosive or negative long run growth rates.

In Barnett and Ghosh (2013), the production function is given by

$$Y_t = H_{Y_t}^{1-\alpha} \int_0^A x(i)^\alpha di, \quad (8.16)$$

where $x(i)$ is the input of intermediate good i , A is the number of available intermediate goods, and $\alpha \in (0,1)$, where $\frac{1}{1-\alpha}$ is the elasticity of substitution for any pair of intermediate goods.

Since research and development (R&D) enable firms to produce new intermediate goods, the R&D technology equation is given by

$$\dot{A} = \gamma H_{A_t}^\lambda A_t^{1-\phi}, \quad \phi > 0, 0 < \lambda \leq 1. \quad (8.17)$$

with

$$H_{A_t} = h_t L_{A_t}, \quad (8.18)$$

where H_{A_t} is effective research effort and A_t is the existing stock of ideas, while ϕ represents the externalities associated with R&D.

In the final goods sector, the representative final output firm rents capital goods, $x(i)$, from monopolist i at price $p(i)$ and pays w as the rental rate per unit of human capital employed. For each durable, the firm chooses quantity $x(i)$ and H_y to maximize the profit as follows:

$$\max_{x, H_y} \int_0^{\infty} [H_y^{1-\alpha} x(i)^\alpha - p(i)x(i)] di - wH_y.$$

Solving the maximization problem gives

$$p(i) = \alpha H_y^{1-\alpha} x(i)^{\alpha-1}, \quad (8.19)$$

$$w = (1 - \alpha) \frac{Y}{H_y}. \quad (8.20)$$

In the intermediate goods sector, each intermediate good, $x(i)$, is produced by a monopolist, who owns an infinitely-lived patent on a technology determining how to transform a unit of raw material, K , costlessly into intermediate goods. That production function is simply $x = K$. The producer of each specialized durable takes $p(i)$ as given from equation (8.19) in choosing the profit maximizing output, x , according to the profit level

$$\pi = \max_x p(x)x - rx,$$

where r is the rental price of raw capital. Solving the monopoly profit maximization problem gives

$$p(i) = \bar{p} = \frac{r}{\alpha}. \quad (8.21)$$

The flow of monopoly profit is

$$\pi(i) = \bar{\pi} = \bar{p}\bar{x} - r\bar{x} = (1 - \alpha)\bar{p}\bar{x}. \quad (8.22)$$

In the research and development sector, the decision to produce a new specialized input depends on a comparison of the discounted stream of net revenue and the cost of the initial investment in a design. Because the market for designs is competitive, the price for designs, P_A , will be bid up until equal to the present value of the net revenue that a monopolist can extract. Therefore P_A is equal to

$$\int_t^{\infty} e^{-\int_t^{\tau} r(s) ds} \pi(\tau) d\tau = P_A(t), \quad (8.23)$$

where r is the interest rate.

If $v(t)$ denotes the value of the innovation, then

$$v(t) = \int_t^{\infty} e^{-\int_t^{\tau} r(s) ds} \pi(\tau) d\tau. \quad (8.24)$$

Assuming free entry into the R&D sector, the zero profit condition is

$$wH_A = P_A \underbrace{\gamma H_A^\lambda A^{1-\phi}}_A. \quad (8.25)$$

Therefore, equation (8.25) can equivalently be written as,

$$wH_A = v\gamma H_A^\lambda A^{1-\phi}. \quad (8.26)$$

Because of the symmetry with respect to different intermediate goods, Barnett and Ghosh (2013) set $K = Ax$. The production function then is

$$Y = (AH_Y)^{1-\alpha} (K)^\alpha. \quad (8.27)$$

Hence, from equation (8.20) and (8.27), it follows that

$$w = (1 - \alpha)A \left(\frac{K}{AH_Y} \right)^\alpha. \quad (8.28)$$

From zero profits in the final goods sector, $\pi = H_Y^{1-\alpha} Ax^\alpha - pAx - wH_Y = 0$; and from equation (8.20), the following equation results

$$Y - wH_Y = pAx = \alpha Y. \quad (8.29)$$

Barnett and Ghosh (2013) note that wages equalize across sectors as a result of free entry and exit.

From the consumers' perspective, the agent's utility maximization problem is

$$\max_{c_t, \varepsilon_t} \frac{\int_t^\infty e^{-(\rho-n)t} [c(\tau)^{1-\sigma} - 1]}{1-\sigma} dt$$

subject to

$$\dot{K} = r_t[K_t + v_t A_t] + w_t H_t - c_t N_t - v_t \dot{A}_t - \dot{v}_t A_t,$$

$$\dot{h}_t = \eta h_t^{\beta_1} (1 - \varepsilon_t)^{\beta_2} - \theta g_A h_t, \text{ and } \varepsilon_t \in [0, 1],$$

where ρ is the subjective discount rate with $\rho > n > 0$, and $\sigma \geq 0$ is the inverse of the intertemporal elasticity of substitution in consumption. Individuals choose consumption, c_t , and the fraction of time devoted to human capital production or to market work, ε_t .

In order to conduct bifurcation analysis, Barnett and Ghosh (2013) derive the following equations, which represent the dynamic equations for the model:

$$\frac{\dot{g}}{g} = \left(\frac{\alpha^2}{\sigma} - 1 \right) m - \frac{\rho}{\sigma} + n + g + d, \quad (8.30)$$

$$\frac{\dot{m}}{m} = \frac{1-\alpha}{\alpha} [-\alpha^2 m + \alpha v + \phi(u - v)], \quad (8.31)$$

$$\frac{\dot{v}}{v} = (1 - \alpha)m + v - g + \left\{ \frac{(1-\alpha)\phi}{\alpha} - 1 \right\} (u - v) - d, \quad (8.32)$$

$$\begin{aligned} \frac{\dot{z}}{z} = \frac{1}{f(\beta_2 - 1)} \left[-z - \theta g_A (\beta_1 - 2) + \alpha v - \beta_2 \frac{zvf}{u} - (1 - \phi)(u - v) - n \right] \\ - (1 - \beta_1)(z - \theta g_A), \end{aligned} \quad (8.33)$$

$$\frac{\dot{f}}{f} = \frac{1+f}{f(\beta_2 - 1)} \left[-z - \theta g_A (\beta_1 - 2) + \alpha v - \beta_2 \frac{zvf}{u} - (1 - \phi)(u - v) - n \right], \quad (8.34)$$

$$\frac{\dot{u}}{u} = z - \theta g_A + n - \phi(u - v) + \frac{1}{f(\beta_2 - 1)} \left[-z - \theta g_A (\beta_1 - 2) + \alpha v - \beta_2 \frac{zvf}{u} - (1 - \phi)(u - v) - n \right]. \quad (8.35)$$

According to Barnett and Ghosh's (2013) Definition 1, a steady state is a balanced growth path with zero growth rate. The steady state $s^* = (g^*, m^*, v^*, z^*, f^*, u^*)$ is derived by solving $\dot{g} = \dot{m} = \dot{v} = \dot{z} = \dot{f} = \dot{u} = 0$. The results are as follows:

$$z^* = \frac{n\theta}{\phi},$$

$$v^* = \frac{\rho - n}{\alpha} + \frac{n\sigma}{\phi\alpha},$$

$$u^* = v^* + \frac{n}{\phi},$$

$$m^* = \frac{v^*}{\alpha} + \frac{n}{\alpha^2},$$

$$g^* = \left(1 - \frac{\alpha^2}{\sigma}\right)m^* + \frac{\rho}{\sigma} - n - d,$$

$$f^* = \frac{u^*}{v^*\beta_2} \left(\frac{\phi\rho}{\theta n} - \frac{(\phi + 1 - \sigma)}{\theta} - (\beta_1 - 1) \right).$$

Barnett and Ghosh (2013) derive the growth rate of technology to be $g_A = \frac{n}{\phi}$. The goal is to examine the existence of codimension 1 and codimension 2 bifurcations in the dynamical system defined by (8.30)-(8.35). The usual way to identify codimension-1 bifurcation is by varying a single parameter, while the usual way to identify codimension-2 bifurcation is by varying 2 parameters.

Barnett and Ghosh (2013) discuss reasons accounting for the occurrence of cyclical behaviors. The economic intuition behind the cycle phenomenon is described as follows. Suppose profits for monopolists increase. Then the price for designs, P_A , is bid up, since the

market for designs is competitive. From (8.26), wages, w , in the R&D sector will rise. Higher wages lead to a shift of labor from output production to the research sector. Furthermore, the technological growth rate, g_A , will rise, if externalities to R&D are present. Assuming a negative effect of technical progress on human capital investment, i.e., $\theta > 0$, human capital accumulation, h_t , declines. According to (8.14) and (8.19), the price falls from a decline of average quality of labor. Monopoly profits then fall, completing the mechanism of this cycle.

Barnett and Ghosh (2013) use the numerical continuation package Matcont to detect Andronov-Hopf bifurcations. Table 8.2 reports the values of the subjective discount rate, ρ , the share of human capital, β_1 , and the share of time devoted to the human capital production, β_2 , the effect of technological progress on human capital accumulation, θ , and the depreciation rate of capital, d . Those parameters are treated as free parameters, at which Hopf bifurcation can occur.²³

As discussed in section 8.2, a positive first Lyapunov coefficient indicates the existence of subcritical Hopf bifurcation. Therefore, since cases reported in Table 8.2 are associated with positive first Lyapunov coefficients, an unstable limit cycle with periodic orbit bifurcates from the equilibrium. When ρ, β_1, θ , and d are treated as free parameters, a slight perturbation of them gives rise to branch points (pitchfork/transcritical bifurcations).

Barnett and Ghosh (2013) investigate the stability properties of cycles generated by different combination of such parameters. The parameter, ρ , taken as a free parameter, gives rise to two period doubling (flip) bifurcations, one of which occurs at $\rho = 0.0257$ and the other at $\rho = 0.0258$. Both bifurcations have negative normal form coefficients, indicating stable double-period cycles.

²³ Table 8.2 is a replicate of Barnett and Ghosh (2013) Table 1.

Table 8.2. Stability Analysis of a Variant of Jones Semi-Endogenous Growth Model

Parameters varied	Equilibrium bifurcation	Continuation
$\beta_1: \{\alpha = 0.4, \rho = 0.055, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 8\}$	Branch Point (BP) $\beta_1 = 1$	
$\beta_1: \{\alpha = 0.4, \rho = 0.025772, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08\}$	Hopf (H) First Lyapunov coefficient=0.0000230, $\beta_1 = 0.19$	
$\beta_2: \{\alpha = 0.4, \rho = 0.025772, \beta_1 = 0.19, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08\}$	Hopf (H) First Lyapunov coefficient=0.00002302, $\beta_2 = 0.040000$	
$d: \{\alpha = 0.4, \beta_1 = 0.19, \rho = 0.055, \beta_2 = 0.04, n = 0.01, \theta = 0.4, \phi = 1, \sigma = 8\}$	Branch Point (BP) $d = 0.826546$	
$\rho: \{\alpha = 0.4, \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 0.08\}$	Hopf (H) First Lyapunov coefficient=0.0000149, $\rho = 0.025772$ Branch Point (BP) $\rho = 0.026726$ Hopf (H) Neutral saddle $\rho = 0.026698$	Bifurcation of limit cycle Period doubling (period=1569.64; $\rho = 0.0257$) Normal form coefficient=-4.056657e-013 Period doubling (period=1741.46; $\rho = 0.0258$) Normal form coefficient=-7.235942e-015 Limit point cycle (period=2119.53; $\rho = 0.0258$) Normal form coefficient=7.894415e-004 Period doubling (period=2132.13; $\rho = 0.0258$) Normal form coefficient=-1.763883e-013
$\theta: \{\alpha = 0.4, \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \rho = 0.029710729, \phi = 0.69716983, \sigma = 0.08\}$	Hopf (H) First Lyapunov coefficient=0.0000230, $\theta = 0.40000$ Hopf (H) First Lyapunov coefficient=0.00001973, $\theta = 0.355216$ Hopf (H) Neutral saddle, $\theta = 0.612624$ Branch Point (BP) $\theta = 0.613596$	Codimension-2 bifurcation Generalized Hopf(GH) $\theta = 0.000044, \rho = 0.580853, L2 = 0.000001254$ Bogdanov-Takens(BT) $\theta = 0, \rho = 0.644247$ $(a, b) = (0.000001642, -0.003441)$ Generalized Hopf(GH) $\theta = 0.000055, \beta_1 = 0.584660, L2 = 0.0000008949$ Bogdanov-Takens(BT) $\theta = 0, \beta_1 = 0.903003$ $(a, b) = (0.000006407790, 0.03291344)$

From further computation, Barnett and Ghosh (2013) find two limit cycles with different periods present near the LPC point at $\rho = 0.0258$ bifurcating from the Hopf point. They also find another period doubling (flip) bifurcation at $\rho = 0.0258$. Barnett and Ghosh (2013) then investigate the existence of codimension-2 bifurcations by first taking θ and ρ as free

parameters and then taking θ and β_1 as free parameters. There are two types of codimension 2 bifurcations: Bogdanov-Takens and Generalized Hopf. At each Bogdanov-Takens point the system has an equilibrium with a double zero eigenvalue. The bifurcation point of the Generalized Hopf bifurcation separates branches of subcritical and supercritical Andronov-Hopf bifurcations in the parameter plane. The Generalized Hopf points are nondegenerate, since the second Lyapunov coefficient is nonzero. The system has two limit cycles for nearby parameter values, which collide and disappear through a saddle–node bifurcation.

9. Conclusion

At this stage of this research, we believe that Grandmont's conclusions appear to hold for all categories of dynamic macroeconomic models, from the oldest to the newest. So far, the findings we have surveyed suggest that Barnett and He's initial findings with the policy-relevant Bergstrom-Wymer model appear to be generic. We anticipate that further studies with other models will produce similar results, and advances in nonlinear and stochastic bifurcation are likely to find even deeper classes of bifurcation behavior, including perhaps chaos, which is precluded by linearization. This survey is designed to facilitate such future studies.

The practical implications of these findings include the following. (1) Policy simulations with macroeconometric models should be run at various points within the confidence regions about parameter estimates, not just at the point estimates. Robustness of dynamical inferences based on simulations only at parameters' point estimates is suspect. (2) Increased emphasis on measurement of variables is warranted, since small changes in variables can alter dynamical inferences by moving bifurcation boundaries and their distances from parameter point estimates. (3) While bifurcation phenomena are well known to growth model theorists, econometricians should take heed of the views of systems theorists, who have found that bifurcation stratification of the parameter space of dynamic systems is normal, and should not be viewed as a source of model failure or defect.

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